Surface Approximation with Triangle Meshes

S U R F A C E S A N D M E S H E S

Outline

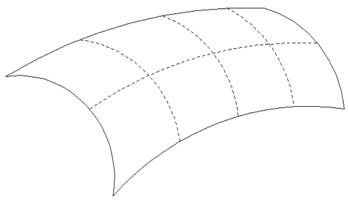
S U R F A C E S A N D M E S H E S

- Classification of surfaces
- Approximating surfaces with triangle meshes
- Encoding triangle meshes
- Compressed mesh representations

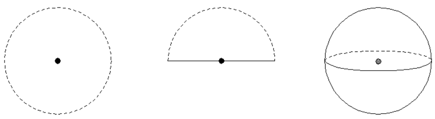
What is a surface?

SURFACES AND MESHES

- A surface ***S*** embedded in space is a subset of ***R^3*** that is intrinsically two-dimensional



- For any neighborhood ***u_P*** of a point ***P*** on ***S***
  - ***u_P*** contains at least half of an open disk (i.e., no part of ***S*** is less than two-dimensional)
  - ***u_P*** does not contain any open ball (i.e., no part of ***S*** is solid)



Surface Modeling

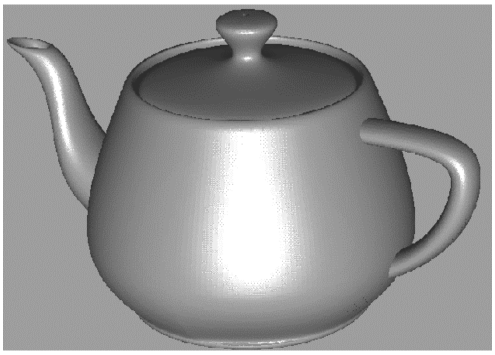
SURFACES AND MESHES

- Surfaces defined in the continuum:
  - Sources: mathematics, CAD
  - Problem: a finite (digital) representation is necessary for surface analysis and rendering
- Surfaces known at a finite set of points:
  - Source: sampling (range scanners, photogrammetry, medical data), simulation (finite element methods)
  - Problem: a surface in the continuum must be defined through a reconstruction process

Topological Characterization of Surfaces

- Manifold without boundary

a surface ***S*** in $\boldsymbol{R^3}$ such that any point on ***S*** has an open neighborhood homeomorphic to an open disk in $\boldsymbol{R^2}$



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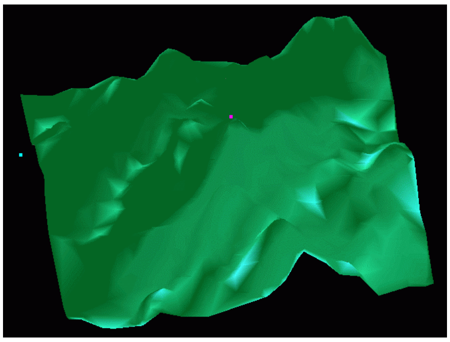
...Topological Characterization of Surfaces...

- Manifold with boundary

a surface ***S*** in $\boldsymbol{R^3}$ such that any point on ***S*** has an open neighborhood homeomorphic to an open disk or to half an open disk in $\boldsymbol{R^2}$



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...Topological Characterization of Surfaces...

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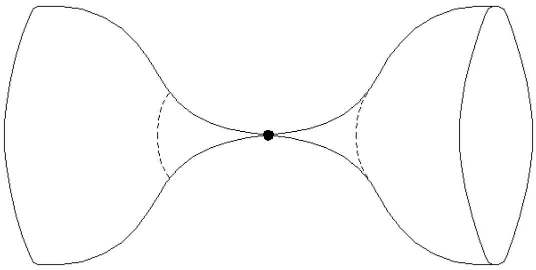
- An example of a non-manifold situation



Geometric Representation of Surfaces

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- Implicit form:

  an *implicit surface* is the locus of solutions of an equation

  F(x,y,z) = 0

  where ***F*** is a mathematical expression of three variables

  Problems:

  - definition is too general: some expressions give objects that are not intrinsically two-dimensional
  - we might not know expression ***F(x,y,z)***
  - even if we know ***F***, we might not be able to solve the equation

  Remark: surfaces which cannot be described in an analytic form are called *free-form* surfaces

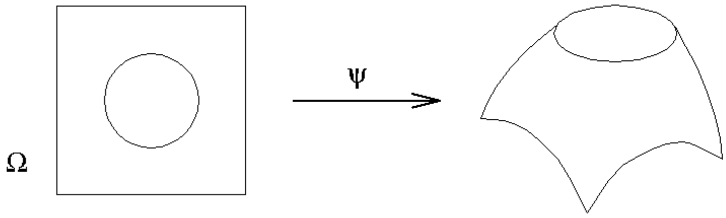
...Geometric Representation of Surfaces...

- Parametric form:

  a *parametric patch* is the image of a continuous function

  $$y : W \subset \boldsymbol{R}^2 \rightarrow \boldsymbol{R}^3$$

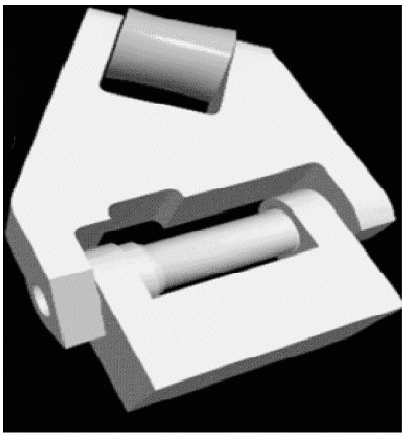
  - **W** is called *parametric space*
  - $\boldsymbol{R}^3$ is called *physical space*
  - boundaries of **W** and of **y (W)** are formed by *trimming curves*



...Geometric Representation of Surfaces...

- Parametric surface:

  a collection of parametric patches properly abutting

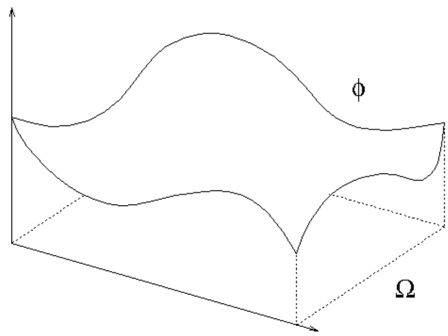


...Geometric Representation of Surfaces...

- Explicit surfaces:
  - Special case of a parametric surface
  - A surface can be represented as a bivariate function when it is the image of a scalar field

$$f : W \subset \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}$$

- Example: topographic surfaces



Hypersurfaces

- Generalization of explicit surfaces to higher dimensions
- Image of a scalar field

$$f : W \longrightarrow \boldsymbol{R}$$

where **W** is a compact domain in $\boldsymbol{R}^k$

- Example: volume data (for ***k=3***)

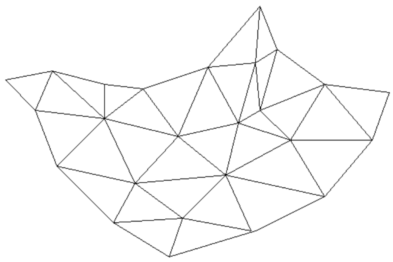
Approximating surfaces with triangle meshes

SURFACES AND MESHES

- Surface representation:

  mesh of triangles (i.e., a set of triangles such that any two of them either do not intersect or share a common edge or vertex)

- Each triangle approximates a surface patch within a given accuracy
- Triangle meshes are easy to represent, manipulate, visualize
- Triangle meshes can be constructed from irregularly sampled data

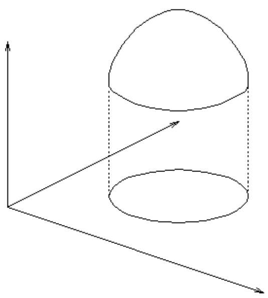


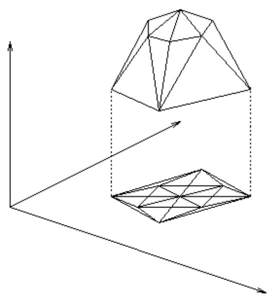
Approximating a 2-dimensional scalar field with a triangle mesh

SURFACES AND MESHES

K=2:

- A 2-dimensional scalar field is described as a function **z = f (x,y)**
- A triangle meshes in 3D is obtained by triangulating the domain of **f** and lifting it to three-dimensional space





Approximating a 3-dimensional scalar field with a tetrahedral mesh

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k=3:

- A 3-dimensional scalar field is defined by a function $z = f(x_1, x_2, x_3)$
- An approximation is obtained by discretizing the domain of **f** with a tetrahedral mesh
- A *tetrahedral mesh* is a set of tetrahedra such that any two of them either do not intersect or share a common face, edge or vertex

Approximating a k-dimensional scalar field with a simplicial mesh

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- Discretization of the domain of a k-dimensional field $z = f(x_1, x_2, \ldots, x_k)$ with a simplicial mesh

$\Downarrow$

- Defines a linear approximation of **f** in (k+1)-dimensional Euclidean space

How to compute the approximation?

- How well does a mesh approximate a given surface?
- We are not given surfaces but
  - *mesh of triangles* for free-form surfaces
  - *set of points* at which the field is known for scalar fields (hypersurfaces)

The Error Metric

- *Euclidean distance* between a point ***p*** and a set $\boldsymbol{Q} \subseteq \boldsymbol{R}^k$:

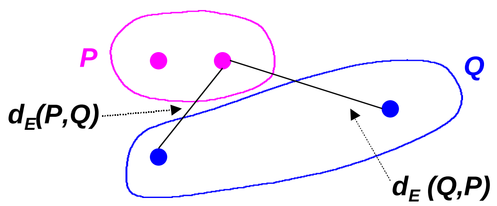
$$d(p,Q) = \inf \{ d(p,q) \mid q \text{ in } Q \}$$

where ***d(p,q)*** is the Euclidean distance between point ***p*** and ***q***

- *"Distance"* from a set ***P*** to a set ***Q***

$$d_E(P,Q) = \sup \{ d(p,Q) \mid p \text{ in } P \}$$

However $d_E(P,Q) <> d_E(Q,P)$



...The Error Metric...

- *Hausdorff distance* defined as:

$$d_H(P,Q) = \max \{ d_E(P,Q), d_E(Q,P) \}$$

It follows that $d_H(P,Q) = 0$ iff $P = Q$

Thus, we can express the distance between a surface ***S*** and its approximating triangle mesh ***T*** as $d_H(S,T)$

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...The Error Metric...

Discrete case

- Free-form surfaces:
  - Surface ***S*** given as a fine mesh of triangles T_S
  - ==> we measure distance between two triangle meshes
- Hypersurfaces:
  - Scalar field **f** is known at a finite set of points ***Q***
  - ==> we measure the distance of the points of ***Q*** from the triangle mesh ***T*** approximating the hypersurface

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The *Metro* tool [Cignoni et al. 98]

Given two triangular meshes T_1 and T_2, ***Metro*** :

- **scan converts** each triangle ***t*** of T_1 with a user-selected scan step, *or*, alternatively, chooses a set ***P*** of **points** distributed **randomly** on ***t***
- for each point ***p*** in ***P***, computes $d(p,T_2)$ (distances are computed efficiently using a bucketing data structure)

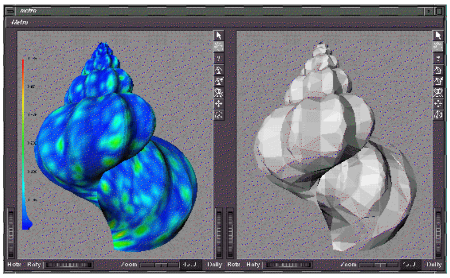
and switches meshes to be symmetric.

- precision of the evaluation depends on *sampling resolution !*
- with a sufficiently fine sampling step, almost equal results in both directions (e.g., 0.01% of mesh bounding box diagonal)

SURFACES AND MESHES

Metro returns

- accurate **numerical** distance estimation
- a **visual** representation of the approximation error



- tool runs on SGI ws (OpenInventor)
- available in public domain

...The Error Metric...

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Approximating the error on scalar fields

- Error of a point ***p*** of set ***Q*** defined as

$$e(p) = | f(p) - f_T(p) |$$

where

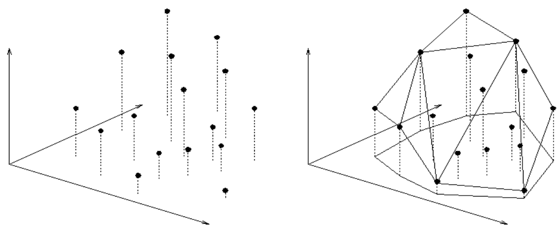
- f *(p)* is the known value of the field at ***p***
- $f_T(p)$ is the approximated value of the field at ***p*** computed on the basis of simplicial mesh ***T***

...The Error Metric...

S U R F A C E S A N D M E S H E S

- Error function defined by a discrete norm:
  - $E(T,Q) = \| e(p) \|$
  - $E(T,Q) = \max \{e(p), p \in Q\}$

- Example: two-dimensional scalar field (terrain)



Remarks

- More accurate representation $\Rightarrow$ more triangles
- More triangles $\Rightarrow$ higher storage and processing time

Tradeoff between accuracy and space / time:

- adapting the accuracy to the needs of an application can improve efficiency
- accuracy might be variable over different portions of the object