Notation

$\mathbf{M}$	Matrix in $\mathbb{R}^{3\times3}$
$\mathbf{v}$	Column vector in $\mathbb{R}^3$
v_i	Components of a vector $\mathbf{v}$
$\begin{pmatrix}\mathbf{a} & \mathbf{b} & \mathbf{c}\end{pmatrix}$	Block matrix of multiple matrices/vectors
$\overline{\mathbf{M}}$, $\overline{\mathbf{r}}$	Matrix in $\mathbb{R}^{6\times6}$/vector in $\mathbb{R}^6$
$\mathbf{j}_i$	Unit vector along i-th coordinate
$\mathbf{M}_x, \mathbf{v}_y, \dots$	Partial derivatives of a Matrix/vector in $x, y, \dots$
∇	Nabla operator $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \dots)^{\mathrm{T}}$

Proof of Theorem 1

Theorem 1 The PEV operator delivers structurally stable curves that are either closed or end at the boundaries of the domain.

The main idea to prove Theorem 1 is to search for PEV lines not in 3D (x,y,z) space but in a 6D (x,y,z,u,v,w) space: at every point $\mathbf{x} = (x,y,z)^{\mathrm{T}}$, all vector directions $\mathbf{r} = (u,v,w)^{\mathrm{T}}$ are checked for being an eigenvector of $\mathbf{S}$ and $\mathbf{T}$. This means that we search for all 6D points $(\mathbf{x},\mathbf{r})^{\mathrm{T}}$ fulfilling $\mathbf{S}(\mathbf{x})\,\mathbf{r}\times\mathbf{r} = \mathbf{0}$ and $\mathbf{T}(\mathbf{x})\,\mathbf{r}\times\mathbf{r} = \mathbf{0}$. We formulate this to search for all 6D points $(\mathbf{x},\mathbf{r})^{\mathrm{T}}$ where a 6D vector field $\overline{\mathbf{h}}$ vanishes:

$$\overline{\mathbf{h}}(\mathbf{x},\mathbf{r}) = \begin{pmatrix}\mathbf{S}(\mathbf{x})\,\mathbf{r}\times\mathbf{r}\\ \mathbf{T}(\mathbf{x})\,\mathbf{r}\times\mathbf{r}\end{pmatrix} = \overline{\mathbf{0}}. \quad (1)$$

Suppose a point $(\mathbf{x}_0,\mathbf{r}_0)^{\mathrm{T}}$ is on a PEV structure, i.e., fulfills (1). In order to study the PEV structures in a linear neighborhood of $(\mathbf{x}_0,\mathbf{r}_0)^{\mathrm{T}}$, we search for all directions $(d\mathbf{x}, d\mathbf{r})^{\mathrm{T}}$ in which $\overline{\mathbf{h}}$ remains zero: $\nabla\overline{\mathbf{h}}\cdot(d\mathbf{x},d\mathbf{r})^{\mathrm{T}} = \overline{\mathbf{0}}$. In other words: we have to explore the null space of $\nabla\overline{\mathbf{h}}$. Applying elementary differentiation rules gives

$$\nabla\overline{\mathbf{h}} = \begin{pmatrix}\mathbf{G}_1 & \mathbf{G}_3\\ \mathbf{G}_2 & \mathbf{G}_4\end{pmatrix}$$

with

$$\mathbf{G}_1 = \begin{pmatrix}\mathbf{S}_x\,\mathbf{r}\times\mathbf{r} & \mathbf{S}_y\,\mathbf{r}\times\mathbf{r} & \mathbf{S}_z\,\mathbf{r}\times\mathbf{r}\end{pmatrix}$$
$$\mathbf{G}_2 = \begin{pmatrix}\mathbf{T}_x\,\mathbf{r}\times\mathbf{r} & \mathbf{T}_y\,\mathbf{r}\times\mathbf{r} & \mathbf{T}_z\,\mathbf{r}\times\mathbf{r}\end{pmatrix}$$
$$\mathbf{G}_3 = \begin{pmatrix}\mathbf{S}\mathbf{j}_1\times\mathbf{r}+\mathbf{S}\mathbf{r}\times\mathbf{j}_1 & \mathbf{S}\mathbf{j}_2\times\mathbf{r}+\mathbf{S}\mathbf{r}\times\mathbf{j}_2 & \mathbf{S}\mathbf{j}_3\times\mathbf{r}+\mathbf{S}\mathbf{r}\times\mathbf{j}_3\end{pmatrix}$$
$$\mathbf{G}_4 = \begin{pmatrix}\mathbf{T}\mathbf{j}_1\times\mathbf{r}+\mathbf{T}\mathbf{r}\times\mathbf{j}_1 & \mathbf{T}\mathbf{j}_2\times\mathbf{r}+\mathbf{T}\mathbf{r}\times\mathbf{j}_2 & \mathbf{T}\mathbf{j}_3\times\mathbf{r}+\mathbf{T}\mathbf{r}\times\mathbf{j}_3\end{pmatrix}.$$

Then

$$\mathbf{G}_1{}^{\mathrm{T}}\mathbf{r} = \mathbf{G}_2{}^{\mathrm{T}}\mathbf{r} = 0 \quad (2)$$

and from (1) follows

$$\mathbf{G}_3{}^{\mathrm{T}}\mathbf{r} = \mathbf{G}_4{}^{\mathrm{T}}\mathbf{r} = 0. \quad (3)$$

and

$$\mathbf{G}_3\,\mathbf{r} = \mathbf{G}_4\,\mathbf{r} = 0. \quad (4)$$

Equations (2) and (3) give that

$$\mathrm{rank}\big(\nabla\overline{\mathbf{h}}\big) = 4 \quad (5)$$

in the structurally stable case. This means that for $\mathrm{rank}\big(\nabla\overline{\mathbf{h}}\big) < 4$, adding noise to $\mathbf{S},\mathbf{T}$ brings $\mathrm{rank}\big(\nabla\overline{\mathbf{h}}\big)$ to 4. Equation (5) means that the PEV structure around $(\mathbf{x}_0,\mathbf{r}_0)^{\mathrm{T}}$ is a 2-manifold in 6D. To see Equation (5), we consider a rotation of the underlying coordinate system such that $\mathbf{r} = (0,0,r_z)$. Then Equations (2) and (3) give that the rotated tensors $\mathbf{G}_1,\mathbf{G}_2,\mathbf{G}_3,\mathbf{G}_4$ have vanishing third columns. This and Equation (1) gives that $\nabla\overline{\mathbf{h}}$ has two colums, which proves Equation (5).

One vector in the null space of $\nabla\overline{\mathbf{h}}$ is trivial and denotes a simple scaling of $\mathbf{r}$: Equations (2) – (4) give $\nabla\overline{\mathbf{h}}\cdot(\mathbf{0},\mathbf{r})^{\mathrm{T}} = \overline{\mathbf{0}}$. This means that the projection of the null space of $\nabla\overline{\mathbf{h}}$ into the spatial subspace $\mathbf{x}$ gives a one-manifold in 3D. This shows that PEV gives line structures in 3D. To show that they are closed, we consider the 6 components of $\nabla\overline{\mathbf{h}}$ as scalar fields and interpret the PEV structure as intersection of their 5D iso-hypersurfaces. Iso-hypersurfaces are always closed, which means their intersections are also closed.

Note that the proof did not make any assumptions on the behavior of $\mathbf{S},\mathbf{T}$ around $(\mathbf{x}_0,\mathbf{r}_0)^{\mathrm{T}}$. This means that it holds also in case of a transition from real to imaginary eigenvectors of $\mathbf{S}$ or $\mathbf{T}$ as well as in regions of isotropic tensors.