Laplacian in one minute

\[ \nabla f(x) = \text{direction of the steepest increase of } f \text{ at } x \]

\[ \text{Divergence } \nabla \cdot F(x) = \text{density of an outward flux of } F \text{ from an infinitesimal volume around } x \]

\[ \text{Divergence theorem: } \int_V \nabla \cdot F \, dV = \int_{\partial V} \langle F, \hat{n} \rangle \, dS \]

\[ \sum \text{sources} + \text{sinks} = \text{net flow} \]

\[ \Delta f(x) = -\nabla \cdot \nabla f(x) \]

Smooth scalar field \( f \)
Laplacian in one minute

- **Gradient** \( \nabla f(x) = \) ‘direction of the steepest increase of \( f \) at \( x \)’

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Laplacian in one minute

- **Gradient** $\nabla f(x) = \text{‘direction of the steepest increase of } f \text{ at } x$’

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Smooth vector field $F$
Laplacian in one minute

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‘\( \sum \) sources + sinks = net flow’

- **Laplacian** \( \Delta f(x) = -\text{div}(\nabla f(x)) \)
  ‘difference between \( f(x) \) and the average of \( f \) on an infinitesimal sphere around \( x \)’ (consequence of the Divergence theorem)

*We define Laplacian with negative sign*
Physical application: heat equation

\[ f_t = -c \Delta f \]

Newton’s law of cooling: rate of change of the temperature of an object is proportional to the difference between its own temperature and the temperature of the surrounding

\( c \ [\text{m}^2/\text{sec}] = \text{thermal diffusivity constant} \) (assumed = 1)
Riemannian geometry in one minute

- **Tangent plane** $T_x X = \text{local Euclidean representation of manifold (surface) } X \text{ around } x$
Riemannian geometry in one minute

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$$\langle \cdot, \cdot \rangle_{T_xX} : T_xX \times T_xX \rightarrow \mathbb{R}$$

depending smoothly on $x$
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  \exp_x : T_x X \rightarrow X
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  ‘unit step along geodesic’

- **Geodesic** = shortest path on $X$ between $x$ and $x'$
Laplace-Beltrami operator

\[ \nabla_X f(x) = \nabla(f \circ \exp_x)(0) \]

Taylor expansion
\[ (f \circ \exp_x)(v) \approx f(x) + \langle \nabla_X f(x), v \rangle_{T_xX} \]

Laplace-Beltrami operator
\[ \Delta_X f(x) = \Delta(f \circ \exp_x)(0) \]

Smooth field \( f : X \to \mathbb{R} \)

\( x \)

\( f \)

Smooth field \( f : X \to \mathbb{R} \)
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Smooth field

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f \circ \exp_x : T_x X \to \mathbb{R}
\]

Intrinsic (expressed solely in terms of the Riemannian metric)

Isometry-invariant

Self-adjoint

Positive semidefinite \(\Rightarrow\) non-negative eigenvalues
Laplace-Beltrami operator

- **Intrinsic gradient**
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  \[ \langle \Delta_X f, g \rangle_{L^2(X)} = \langle f, \Delta_X g \rangle_{L^2(X)} \]
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- **Positive semidefinite** \( \Rightarrow \) non-negative eigenvalues
Discrete Laplacian (Euclidean)

One-dimensional

\[(\Delta f)_i \approx 2f_i - f_{i-1} - f_{i+1}\]

Two-dimensional

\[(\Delta f)_{ij} \approx 4f_{ij} - f_{i-1,j} - f_{i+1,j} - f_{i,j-1} - f_{i,j+1}\]
Discrete Laplacian (non-Euclidean)

Undirected graph \((V, E)\)

\[
(\Delta f)_i \approx \sum_{(i,j) \in E} w_{ij}(f_i - f_j)
\]

Triangular mesh \((V, E, F)\)

\[
(\Delta f)_i \approx \frac{1}{a_i} \sum_{(i,j) \in E} \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2} (f_i - f_j)
\]

\(a_i = \text{local area element}\)

Tutte 1963; MacNeal 1949; Duffin 1959; Pinkall, Polthier 1993
Physical application: heat equation

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**Newton’s law of cooling:** rate of change of the temperature of an object is proportional to the difference between its own temperature and the temperature of the surrounding

\[ c \, [m^2/sec] = \text{thermal diffusivity constant} \, (\text{assumed} = 1) \]
A function \( f : [-\pi, \pi] \rightarrow \mathbb{R} \) can be written as Fourier series

\[
f(x) = \sum_{\omega} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi) e^{i\omega \xi} d\xi \quad e^{-i\omega x}
\]

\[
\hat{f}(\omega) = \langle f, e^{-i\omega x} \rangle_{L^2([-\pi, \pi])}
\]

Fourier basis = Laplacian eigenfunctions:

\[
\Delta e^{-i\omega x} = \omega^2 e^{-i\omega x}
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$$= \alpha_1 + \alpha_2 + \alpha_3 + \ldots$$
Fourier analysis (Euclidean spaces)

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\[
\begin{align*}
\text{Fourier basis} &= \text{Laplacian eigenfunctions: } \Delta e^{-i\omega x} = \omega^2 e^{-i\omega x} \\
&= \alpha_1 + \alpha_2 + \alpha_3 + \ldots
\end{align*}
\]
Fourier analysis (non-Euclidean spaces)

A function $f : X \to \mathbb{R}$ can be written as Fourier series

$$f(x) = \sum_{k \geq 1} \int_X f(\xi) \phi_k(\xi) d\xi \quad \phi_k(x)$$

$$\hat{f}_k = \langle f, \phi_k \rangle_{L^2(X)}$$

$$\phi_1 = \alpha_1 + \alpha_2 + \alpha_3 + \ldots$$

Fourier basis = Laplacian eigenfunctions: $\Delta_X \phi_k(x) = \lambda_k \phi_k(x)$
Convolution (Euclidean spaces)

Given two functions $f, g : [-\pi, \pi] \to \mathbb{R}$ their convolution is a function

$$(f \ast g)(x) = \int_{-\pi}^{\pi} f(\xi)g(x - \xi)\,d\xi$$

Convolution Theorem: Fourier transform diagonalizes the convolution operator

$$f \ast g = F^{-1}(Ff \cdot Fg)$$

d’Alembert 1754; Borel 1899
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Convolution Theorem: Fourier transform diagonalizes the convolution operator $\Rightarrow$ convolution can be computed in the Fourier domain as

$$f \ast g = \mathcal{F}^{-1}(\mathcal{F}f \cdot \mathcal{F}g)$$

d’Alembert 1754; Borel 1899
Convolution (non-Euclidean spaces)

Generalized convolution of \( f, g \in L^2(X) \) can be defined by analogy

\[
(f \ast g)(x) = \sum_{k \geq 1} \langle f, \phi_k \rangle_{L^2(X)} \langle g, \phi_k \rangle_{L^2(X)} \phi_k(x)
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product in the Fourier domain

Not shift-invariant!

Represent filter in the Fourier domain

Problem: Filter coefficients depend on basis $\{\phi_k\}_{k \geq 1}$

⇒ does not generalize to other domains!

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- Product in the Fourier domain
- Inverse Fourier transform

Shuman et al. 2013
Convolution (non-Euclidean spaces)

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- **Not shift-invariant!**
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- Not shift-invariant!
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Convolution (non-Euclidean spaces)

Function $f$

Filtered function $\tilde{f}$

Henaff, Bruna, LeCun 2015
Convolution (non-Euclidean spaces)

Function $f$

Filtered function $f$

Same filter another shape

Henaff, Bruna, LeCun 2015
Heat diffusion on manifolds

\[
\begin{cases}
    f_t(x, t) = -\Delta_X f(x, t) \\
    f(x, 0) = f_0(x)
\end{cases}
\]

- \( f(x, t) \) = amount of heat at point \( x \) at time \( t \)
- \( f_0(x) \) = initial heat distribution
Heat diffusion on manifolds

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- \(f(x, t) = \text{amount of heat at point } x \text{ at time } t\)
- \(f_0(x) = \text{initial heat distribution}\)

Solution of the heat equation expressed through the heat operator

\[
f(x, t) = e^{-t\Delta_x} f_0(x)
\]
Heat diffusion on manifolds

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\left\{ \begin{array}{l}
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Solution of the heat equation expressed through the heat operator

\[
f(x, t) = e^{-t\Delta_X} f_0(x) = \sum_{k \geq 1} \langle f_0, \phi_k \rangle_{L^2(X)} e^{-t\lambda_k} \phi_k(x)
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Heat diffusion on manifolds

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 &= \int_X f_0(\xi) \sum_{k \geq 1} e^{-t\lambda_k} \phi_k(x) \phi_k(\xi) \, d\xi \\
 &= \underbrace{\text{heat kernel } h_t(x, \xi)}
\end{align*}
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- “impulse response” to a delta-function at $\xi$
Heat diffusion on manifolds

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\end{aligned}
\]

- “impulse response” to a delta-function at \( \xi \)
- “how much heat is transferred from point \( x \) to \( \xi \) in time \( t \)"
Heat kernels at different points (rows/columns of matrix $e^{-t\Delta x}$)
Autodiffusivity

Autodiffusivity = diagonal of matrix $e^{-t\Delta x}$

Related to Gaussian curvature by virtue of the Taylor expansion

$$h_t(x, x) \approx \frac{1}{4\pi t} + \frac{K(x)}{12\pi} + O(t)$$

Sun, Ovsjanikov, Guibas 2009
Spectral descriptors

\[ f(x) = \sum_{k \geq 1} \begin{pmatrix} \tau_1(\lambda_k) \\ \vdots \\ \tau_Q(\lambda_k) \end{pmatrix} \phi_k^2(x) \]

Heat Kernel Signature (HKS)

\[ \tau_i(\lambda) = \exp(-\lambda t_i) \]

Heat autodiffusivity

Wave Kernel Signature (WKS)

\[ \tau_i(\lambda) = \exp\left(-\frac{(\log e_i - \log \lambda)^2}{\sigma^2}\right) \]

Band-pass filter bank

Probability of a quantum particle

Sun, Ovsjanikov, Guibas 2009; Aubry, Schlickewei, Cremers 2011