Chapter 3: Clifford analysis

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3.1 Motivation for Differential Calculus

We know several important differential operators. We begin with a $C^1$-map.

\[ \varphi : \mathbb{R}^3 \to \mathbb{R} \]

\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \varphi (x_1, x_2, x_3) \]

We know the gradient with the short notation

\[ \nabla \varphi = \left( \frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}, \frac{\partial \varphi}{\partial x_3} \right) \]

The gradient describes the direction with the greatest rate of increase at \( P = (x_1, x_2, x_3)^T \).

A related operator is the directional derivative. For our map \( \varphi \) it is defined by

\[ \partial_b \varphi : \mathbb{R}^3 \to \mathbb{R} \]

\[ s \to \lim_{t \to 0} \frac{1}{t} \varphi(t \mathbf{b} + \mathbf{r}) \]

\[ \partial_b \varphi (r) = \nabla \varphi \cdot \mathbf{b} \]

We know the gradient

\[ \nabla \varphi : \mathbb{R}^3 \to \mathbb{R}^3 \]

\[ \begin{bmatrix} \frac{\partial \varphi}{\partial x_1} \\ \frac{\partial \varphi}{\partial x_2} \\ \frac{\partial \varphi}{\partial x_3} \end{bmatrix} \]

with the short notation

\[ \nabla \varphi = \nabla \varphi \]

\[ \partial_b \varphi (r) \] describes the rate of change of \( \varphi \) in direction \( \mathbf{b} \).
For a vector field
\[ v \colon \mathbb{R}^3 \rightarrow \mathbb{R}^3 \]
there are two important derivatives. The divergence is the first one:
\[ \text{div } v \colon \mathbb{R}^3 \rightarrow \mathbb{R} \]
\[ f \rightarrow \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \]
It has the short notation
\[ \text{div } v = \nabla \cdot v \]
and measures the outflow of an infinitesimal volume \( V \) centered at \( P \) per unit volume.

The other differential operator is the curl.
\[ \text{curl } v \colon \mathbb{R}^3 \rightarrow \mathbb{R}^3 \]
\[ f \rightarrow \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ \frac{\partial v_3}{\partial x_2} & -\frac{\partial v_3}{\partial x_1} & \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \\ \frac{\partial v_2}{\partial x_3} & \frac{\partial v_1}{\partial x_3} & \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \end{vmatrix} \]
It has the short notation
\[ \text{curl } v = \nabla \times v \]
The vector \( \text{curl } v \) describes the direction of a rotation axis. This axis is perpendicular to the plane where the ratio of circulation around the boundary of an area segment and the area of the segment takes its maximum.

Goal: We want to unify all this operators into one which is independent of any coordinate system.

3.2 Differential Calculus in 3D
For a coordinate invariant derivative we need the notation of reciprocal vectors in three dimensions. Let
\[ \{ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \} \subset \mathbb{R}^3 \subset \mathcal{G}_3 \]
be a basis. Then one defines reciprocal vectors
\[ \{ \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3 \} \subset \mathbb{R}^3 \subset \mathcal{G}_3 \]
The following rules hold:

\[ A \beta_1 b_1 r() + A \beta_2 b_2 r() = A b r() + A r b() \]

and:

\[ \partial A r() = \sum_{k=1}^{3} g_k A g_k r() \]

where \((e^1, e^2, e^3)\) is a basis and \((e_1, e_2, e_3)\) is the reciprocal basis. The element \(A\) can be shown that \(\partial A r()\) is independent of the chosen basis \((e^1, e^2, e^3)\).

As example, let us look at a scalar field \(A: \mathbb{R}^3 \rightarrow \mathbb{R}\) and a surface \(S\) with parametrization \(\phi: \mathbb{R}^2 \rightarrow \mathbb{R}^3\). Then we can calculate the derivative of \(A\) with respect to \(S\), which is given by:

\[ A'(\phi(u_1, u_2)) = \frac{\partial A r()}{\partial u_1} e_1 + \frac{\partial A r()}{\partial u_2} e_2 \]
Consider a point \( P = (x_1, x_2, x_3) \in \mathbb{R}^3 \) and let
\[
\begin{align*}
\vec{e}_1 &= \frac{\partial}{\partial x_1}, & \vec{e}_2 &= \frac{\partial}{\partial x_2}, & \vec{e}_3 &= \vec{e}_1 \times \vec{e}_2 = \vec{n}
\end{align*}
\]

It holds
\[
\frac{\partial \phi}{\partial x} = \sum_{k=1}^3 \vec{e}_k \frac{\partial \phi}{\partial x_k} = \nabla \phi
\]

We have the following relation
\[
\int \text{div} \, v \, dV = \int_S n \cdot \text{div} \, v \, dA
\]

It states that for an arbitrary volume in an application the sum of the divergence in the volume is the net outflow through the surface.

Another important relation is Stokes theorem. It states
\[
\int_{\partial V} \text{curl} \, v \, dS = \int_S v \times dA
\]

with the same notations as before and relates the sum of the curl inside the volume to the flow on the surface.
It is better known in the following case. Let $S \subset \mathbb{R}^3$ be a compact, orientable, piecewise smooth surface with oriented boundary curve $C$. Further, let $n \in \mathbb{R}^3$ be the unit normal in accordance with the right-hand rule. Then holds

$$\int_S \text{curl} \ v \cdot dA = \int_C v \cdot ds.$$ 

If $v$ describes a force acting on particles, the theorem will state that the total work done on a particle traveling on $C$ equals the integral of the curl on the surface.

$$\int_C v \cdot ds = \int_{\mathbb{R}} \text{curl} \ v \cdot dA.$$ 

3.4 Integration in 3D

We want to introduce now the integration of multivector fields. Let $M \subset \mathbb{R}^3$ be a smooth curve, surface or volume. Let $A : M \to G_1$, $B : M \to G_2$ be two piecewise continuous multivector fields.

Then we define the integral

$$\int_M A dX B$$

as the limit of

$$\lim_{n \to \infty} \sum_{k=1}^n A (x_k) \Delta X (x_k) B (x_k)$$

where $\Delta X (x_k)$ is a curve-, surface- or volume-segment centered at $x_k$ with a measure in the usual Riemannian sense.

In most practical cases the set $M$ is given by a parametrization. Let $r : \mathbb{R} \supset J \to M \subset \mathbb{R}^3$ be a smooth curve. Then we have

$$\int_M A dX B = \int_J A (r (u)) \, du \, g (u) \, B (r (u)) ,$$

where $g (u) = \frac{dr}{du} (u)$.

For a smooth surface patch $r : \mathbb{R}^2 \supset J_1 \times J_2 \to M \subset \mathbb{R}^3$ we get

$$\int_M A dS B = \int_{J_1} \int_{J_2} A (r (u, v)) \, du \, g_1 (u) \, \hat{g} (v) \, B (r (u, v)) ,$$

with

$$g_1 (u_1, u_2) = \frac{\partial}{\partial u_1} (u_1, u_2)$$

and

$$\hat{g} = \frac{\partial}{\partial u_2} (u_1, u_2).$$
Analogous we have for a volume patch
\[ r : \mathbb{R}^3 \to M \subset \mathbb{R}^3 \]
\[(u_1, u_2, u_3) \mapsto r(u_1, u_2, u_3)\]
the definition
\[ \mathbb{R}^3 J_1 J_2 J_3 \times \ldots \times M \mathbb{R} \to u_1 u_2 u_3, \ldots (u_1, u_2, u_3) \]
\[ = A X B d\]
\[ M = \int_A r(\cdot) V r(\cdot) Br(\cdot) d\]
\[ \mathbb{R}^3 \to u_1 u_2 u_3, \ldots (u_1, u_2, u_3) \]

Two important theorems show the relation of the vector derivative and the integral.
Let \( V \subset \mathbb{R}^3 \) be a compact oriented volume with boundary \( \partial V \) and outer unit normal \( n \). \( n^2 = 1 \). Let \( A, B \) be two multivector fields on \( V \). Then we have the fundamental theorem for a compact volume
\[ \int_V A \cdot \partial A = \int \partial V \wedge A \]
where the dots stand for taking the derivative of both fields.

Let us examine this for a vector field \( v \):
We have
\[ v : \mathbb{R}^3 \to \mathbb{R}^3 \]
\[ \mathbb{R}^3 \subset \partial \mathbb{R}^3 \]
\[ = S B \cdot\partial A = \int S v \cdot\partial A \]
\[ \mathbb{R}^3 \times (\cdot) \cdot (\cdot) d\]
\[ \mathbb{R}^3 \times (\cdot) \cdot (\cdot) d\]
\[ \mathbb{R}^3 \times (\cdot) \cdot (\cdot) d\]

We see by comparing the parts of different grades
\[ \int_V \partial v = \int_S (\cdot) \cdot (\cdot) \]
the divergence theorem from Gauss
\[ \int_V \text{curl } v = \int_S (\cdot) \cdot (\cdot) \]
the theorem of Stokes for volumes.

If we start with a compact oriented surface \( S \subset \mathbb{R}^3 \) with boundary \( \partial S \) and a unit normal \( n \), \( n^2 = 1 \), we can prove the fundamental theorem for a compact surface
\[ \int_S A \cdot (\partial v) = \int S B \cdot A \]
\[ \int_S (\cdot) \cdot (\cdot) \cdot (\cdot) d\]
\[ \int_S (\cdot) \cdot (\cdot) \cdot (\cdot) d\]
\[ \int_S (\cdot) \cdot (\cdot) \cdot (\cdot) d\]

If we analyse it for the vector field \( v \), we get
\[ \int_S (\cdot) \cdot (\cdot) \cdot (\cdot) \]
\[ \int_S (\cdot) \cdot (\cdot) \cdot (\cdot) \]
\[ \int_S (\cdot) \cdot (\cdot) \cdot (\cdot) \]
which we may identify as the theorem of Stokes
\[ \int_S (\cdot) \cdot (\cdot) \cdot (\cdot) \]
\[ \int_S (\cdot) \cdot (\cdot) \cdot (\cdot) \]
and the equation

\[ \int_S ((n \times \hat{d}) \cdot v) = \int_{S'} d \times v. \]

which is not so well-known.

In this way we see that Clifford analysis also helps to unify important theorems from integration theory for applications.