Appendix: Curve case

1. Unified interpolatory and approximation curve subdivision

Given the original control vertices $P_i^0 (i = 1, 2, ..., n)$, the unified scheme defines points at level $j + 1$ of the recursion by:

$$
\begin{align*}
P_{2i+1}^{j+1} &= \frac{\alpha - 4}{8} P_i^j + \frac{\alpha}{4} P_i^{j+1} + \frac{1}{8} P_i^{j+1} \\
P_{2i+2}^{j+1} &= \frac{\alpha - 4}{8} P_i^j + \frac{\alpha}{4} P_i^{j+1} + \frac{1}{8} P_i^{j+1}
\end{align*}
$$

(1)

When $\alpha = 0$, the subdivision scheme produces cubic B-splines in the limit. On the other hand, when $\alpha = 1$, the limit curve interpolates the control vertices after several refinements.

Given the original control vertices $P_i^0$ and their weights $\alpha_i^0 (i = 1, 2, ..., n)$, the non-uniform unified scheme defines points at level $j + 1$ of the recursion by:

$$
\begin{align*}
\alpha_{2i}^{j+1} &= \alpha_i \\
\alpha_{2i+1}^{j+1} &= \frac{1}{2} (\alpha_i + \alpha_{i+1}) \\
\alpha_{2i+2}^{j+1} &= \frac{\alpha - 4}{8} P_i^{j+1} + \frac{\alpha}{4} P_i^{j+1} + \frac{1}{8} P_i^{j+1}
\end{align*}
$$

(2)

2. Convergence and smoothness analysis

A well-known method for analyzing convergence and smoothness of curve subdivision schemes has been presented by Dyn and Levin [Dyn02]. Here, we used the method to proof that our uniform unified scheme is $C^4$, and proof that our non-uniform unified stationary scheme is $C^4$ when $\alpha \in [0,1]$.

2.1. Analysis of univariate uniform unified schemes

Following the framework of [Dyn02], a binary subdivision scheme is a convergent if there is a function $p : \mathbb{R} \to \mathbb{R}$ with the property that for any compact set $K \subset \mathbb{R}$, $\lim_{k \to \infty} \max_{i \in K} |p_i^k - p(2^{-k}i)| = 0$.

The refinement rule in equation (1) at refinement level $k$ can be written of the form:

$$p_i^{k+1} = \sum_{j \in \mathbb{Z}} \alpha_{i-2j} p_i^k$$

The symbol of $S_a$ is the Laurent polynomial

$$a(z) = \sum_{i} \alpha_i z^i = \frac{\alpha - 4}{16} z^3 + \frac{\alpha}{4} z^2 + \frac{9 - \alpha}{16} z + \frac{4 - \alpha}{16}$$

This can be written as

$$a(z) = \frac{1}{2} (1 + z^2) b(z),$$

where

$$b(z) = \frac{\alpha - 1}{4} z^3 + \frac{3 - \alpha}{4} z^2 + \frac{1 - \alpha}{4} z + \frac{\alpha - 1}{4}.$$

By Corollary of [Dyn02], if $S_a$ is contractive then $S_a$ is $C^4$. Defining

$$\| S_k^{[l]} \|_{\infty} = \max \left\{ \sum_{i=0}^{2^l} |b_i^{[l]}| : 0 \leq i < 2^l \right\},$$

where $b_i^{[l]} (z) := b(z) b(z^2) \cdots b(z^{2^l})$. we find that

$$\| S_k^{[l]} \|_{\infty} = \max \left\{ \frac{\alpha - 1}{4} + \frac{3 - \alpha}{4} + \frac{1 - \alpha}{4} + \frac{\alpha - 1}{4} \right\} = \frac{1}{2} < 1,$$

which shows that $S_a$ is contractive. So, $S_a$ is $C^4$.

2.2. Analysis of non-uniform unified schemes

A non-uniform linear binary subdivision scheme can be represented by a bi-infinite sequence of generating polynomials $\{f_{i,j,k}(z)\}$ where each $f_{i,j,k}$ is the polynomial representing the scheme generating $p_i^k$, $k \geq 1, j \in \mathbb{Z}$. That is,

$$p_i^{k+1} = \sum_{j \in \mathbb{Z}} f_{i,j,k+1} p_i^k$$

where $f_{i,j,k}(z) = \sum_{m \in \mathbb{Z}} f_{i,j,k,m} z^m$. 
Following the framework of [Dyn02], define \( \delta_{(j,k)}^0(z) \equiv f_{(j,k)}(z) \). Then all the non-uniform difference schemes \( \{\delta_{(j,k)}^r(z)\}, r = 1, \ldots, m \) defined recursively by
\[
\delta_{(j,k)}^r(z) = 2^r (z\delta_{(j-1,k)}^r(z) - \delta_{(j,k)}^r(z))\big/(z^2 - 1)
\]
are finite Laurent polynomials. The generating polynomials of the \( \ell \)-iterated scheme, transforming values at level \( k \) directly to level \( k + \ell \), are \( \{f_{(j,k)}(z)\} \) defined recursively by
\[
f_{(j,k)}(z) = \sum_m q_{(j,k),(m)} z^m q_{(j,k)}(z^2)
\]
where \( q_{(j,k),(m)}(z) = q_{(j,k+1)}(z) \)
and \( f_{(j,k)}(z) = \sum_m f_{(j,k),(m)} z^m, i = 1, \ldots, \ell. \)

In the non-uniform scheme defined by equation (2), the corresponding generating polynomials can be written as
\[
f_{(j,k)}(z) = \frac{1}{2}(1 + z^2) \left( \frac{\alpha_j^k - 1}{8} z^{-1} + \frac{1}{4} z^2 + \frac{3 - \alpha_j^k}{4} z^{-1} + \frac{1}{4} z^0 + \frac{\alpha_j^k - 1}{8} z^1 \right)
\]
\[
\delta_{(j,k)}^0(z) = \alpha_j^k \frac{z^{-1}}{8} + \frac{1 + \alpha_j^k - \alpha_j^{k+1} z^{-2}}{4} + \frac{\alpha_j^k - 1}{8} z^1 + \frac{6 + \alpha_j^k - 4\alpha_j^{k+1} + \alpha_j^{k+2} z^{-1} - \alpha_j^{k+1} - \alpha_j^{k+2} z^0}{4}
\]
\[
\delta_{(j,k),(2)}(z) = \sum_{m=9} q_{m} z^m = \left( \frac{(\alpha_j^{k+2} - 1)\alpha_j^{k+1} - 1}{64} z^9 + \frac{1 + \alpha_j^{k+2} - \alpha_j^{k+1}}{32} (\alpha_j^{k+1} - 1) z^8 + \frac{(6 + \alpha_j^{k+2} - 4\alpha_j^{k+1} + \alpha_j^{k+2}) (\alpha_j^{k+1} - 1)}{64} z^7 + \frac{(\alpha_j^{k+2} - 1) (1 + \alpha_j^{k+1} - \alpha_j^{k+2}) (\alpha_j^{k+1} - 1)}{32} z^6 \right)
\]
\[
\alpha_j^{k+2} - 1(n + \alpha_j^{k+1} - 4\alpha_j^{k+1} + \alpha_j^{k+2}) + \frac{(\alpha_j^{k+2} - 1)(6 + \alpha_j^{k+1} - 4\alpha_j^{k+1} + \alpha_j^{k+2})}{64} + \frac{(\alpha_j^{k+2} - 4\alpha_j^{k+1} + \alpha_j^{k+2})(1 + \alpha_j^{k+1} - \alpha_j^{k+2})}{32} + \frac{1 + \alpha_j^{k+2} - \alpha_j^{k+1} - 1(n + \alpha_j^{k+1} - 4\alpha_j^{k+1} + \alpha_j^{k+2})}{64} + \frac{(\alpha_j^{k+2} - 1)(1 + \alpha_j^{k+1} - \alpha_j^{k+2})}{32}
\]
\[
\alpha_j^{k+2} - 1(n + \alpha_j^{k+1} - 4\alpha_j^{k+1} + \alpha_j^{k+2}) + \frac{(\alpha_j^{k+2} - 1)(6 + \alpha_j^{k+1} - 4\alpha_j^{k+1} + \alpha_j^{k+2})}{64} + \frac{(\alpha_j^{k+2} - 4\alpha_j^{k+1} + \alpha_j^{k+2})(1 + \alpha_j^{k+1} - \alpha_j^{k+2})}{32} + \frac{(\alpha_j^{k+2} - 1)(6 + \alpha_j^{k+1} - 4\alpha_j^{k+1} + \alpha_j^{k+2})}{64} + \frac{(\alpha_j^{k+2} - 1)(6 + \alpha_j^{k+1} - 4\alpha_j^{k+1} + \alpha_j^{k+2})}{64}
\]
\[
\begin{align*}
&+ \frac{(6 + \alpha_{j+2} - 4 \alpha_{j+1} + \alpha_{j+2})(\alpha_{j+1} - 1)}{64} z^1 \\
&+ \frac{(1 - \alpha_{j+1} - \alpha_{j+2})(\alpha_{j+1} - 1)}{64} z^2 \\
&+ \frac{(\alpha_{j+2} - 1)(\alpha_{j+2} - 1)}{64} z^3
\end{align*}
\]

where \( i = \lfloor j/2 \rfloor \). By Corollary of [Dyn02], to check if the scheme \( \{ \tilde{f}_{j,k}(z) \} \) is \( C^1 \) it is enough to prove that \( \{ \tilde{f}_{j,k}(z) \} \) is contractive, and it is enough to show that there exists \( \rho \in [0,1) \) such that each of the following four inequalities holds for any \( j \) and \( k \):

\[
|q_j| + |q_{j+1}| + |q_{j+2}| \leq \rho \quad , \quad |q_{j+1}| + |q_{j+2}| + |q_{j+3}| \leq \rho \quad , \\
|q_{j+2}| + |q_{j+3}| + |q_{j+4}| \leq \rho \quad , \quad |q_{j+3}| + |q_{j+4}| + |q_{j+5}| \leq \rho .
\]

To simplify the notation let us denote the parameters in (5) as \( t_m = \alpha_{j+1}, m = 1, \ldots, 5 \) and \( t_6 = \alpha_{j+2}, m = 6, 7, 8 \). Now we have to show that above four inequalities are satisfied for some fixed \( \rho \in [0,1) \) for any \( \alpha_j \in [0,1] \). The four inequalities take the form:

\[
\frac{1}{64} \left[ (t_k - 1)(4 + 2t_k - 4t_k + 2t_l) + (t_k - 1)(4 + 2t_k - 4t_k + 2t_l) \right] \leq \rho \quad (6)
\]
\[
\frac{1}{64} \left[ (1 + t_k - t_l)(4 + 2t_k - 4t_k + 2t_l) \right] \leq \rho
\]
\[
\frac{1}{64} \left[ (1 + t_k - t_l)(2 + t_k - 2t_k - t_l) \right] \leq \rho
\]
\[
\frac{1}{32} \left[ (1 + t_k - t_l)(4 + 2t_k - 4t_k + 2t_l) \right] \leq \rho \quad (7)
\]
\[
\frac{1}{32} \left[ (1 + t_k - t_l)(2 + t_k - 2t_k - t_l) \right] \leq \rho
\]
\[
\frac{1}{64} \left[ (t_k - 1)(4 + 2t_k - 4t_k + 2t_l) + (t_k - 1)(4 + 2t_k - 4t_k + 2t_l) \right] \leq \rho \quad (8)
\]
\[
\frac{1}{32} \left[ (1 + t_k - t_l)(4 + 2t_k - 4t_k + 2t_l) \right] \leq \rho \quad (9)
\]

The inequalities (6)(7)(8) and (9) are easy satisfied with \( \rho = \frac{7}{8} \) for any \( t_m \in [0,1] \).

Thus, the scheme \( \{ \tilde{f}_{j,k}(z) \} \) is \( C^1 \).

Figure 1. Curve samples of our unified schemes with different value of parameter. From the inside curve to the outside curve where \( \alpha = 1; \alpha = 0.5; \alpha = 0.2; \alpha = 0 \) respectively. The first figure is subdivided once, the others are subdivided quartic. The right two are samples forcing the limit surface to go through a particular set of control points whose \( \alpha_j^0 = 1 \).