On Transfinite Barycentric Coordinates

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Abstract
A general construction of transfinite barycentric coordinates is obtained as a simple and natural generalization of Floater’s mean value coordinates [Flo03, JSW05b]. The Gordon-Wixom interpolation scheme [GW74] and transfinite counterparts of discrete harmonic and Wachspress-Warren coordinates are studied as particular cases of that general construction. Motivated by finite element/volume applications, we study capabilities of transfinite barycentric interpolation schemes to approximate harmonic and quasi-harmonic functions. Finally we establish and analyze links between transfinite barycentric coordinates and certain inverse problems of differential and convex geometry.

1. Introduction
Design and analysis of transfinite interpolation schemes has been a core research topic in geometric modeling since seminal works of CAGD pioneers [Far02]. More recent interest in transfinite interpolation stems from inventing transfinite versions of classical and modern barycentric coordinates [JSW05b, WSHD06, HF06, SJW06].

The contribution of this paper is threefold. First, we invent a general construction of transfinite barycentric coordinates as a weighted version of transfinite mean value coordinates [Flo03, JSW05b] (Section 2). Next, we analyze relationships between transfinite barycentric interpolation schemes and PDE-based interpolation (Sections 3, 4, 5). A special attention is paid to approximating harmonic functions (Sections 3, 4). In particular, we study and generalize simple and elegant transfinite barycentric coordinates proposed twelve years ago by Gordon and Wixom [GW74] and now almost forgotten (Sections 4, 5). Finally, we reveal interesting links between the barycentric interpolation schemes and the famous Christoffel-Minkowski problems studied in differential and convex geometry (Section 6).

After this paper was submitted, we became aware of a recent study [SJW06] where a similar approach to a general construction of 2D transfinite barycentric coordinates was proposed.

While our work is pure theoretical, we believe that the presented results may find applications in computational mechanics and fluid dynamics where numerical methods based on barycentric interpolation schemes gain more and more popularity [AO06, SM06, KFCO06]. Free-form shape deformations constitute another potential area of applications [JSW05b, DM06].

The starting point for our study is a general construction of transfinite barycentric coordinates introduced in [WSHD06]. Given a convex domain Ω, let us consider a smooth function b(x, y), x ∈ Ω and y ∈ ∂Ω, satisfying the following three properties

Non-negativity b(x, y) ≥ 0,                  (1)
Partition of unity ∫∂Ω b(x, y) ds_y = 1,     (2)
Linear precision ∫∂Ω y b(x, y) ds_y = x.     (3)

Now interpolation of function f(y) defined on ∂Ω into Ω is given by

f(x) = ∫∂Ω f(y) b(x, y) ds_y.                  (4)

2. Weighted mean value coordinates
The main idea behind the Floater mean value coordinates [Flo03] consists of applying the mean value property for harmonic functions to piecewise linear functions. Consider a convex bounded domain Ω in R^n and a function f(x) harmonic inside Ω

Δf ≡ ∂²f/∂x₁² + … + ∂²f/∂xₙ² = 0.
Let \( x \) be a point inside \( \Omega \) and \( S(x, r) \) be the sphere of radius \( r \) centered at \( x \) and contained inside \( \Omega \). Then according to the mean value theorem for harmonic functions

\[
f(x) = \frac{1}{|S(x, r)|} \int_{S(x, r)} f(z) \, d\nu.
\]

The authors of [JSW05b] also introduced an elegant continuous version of the mean value coordinates. Below we give a simple derivation of the continuous version of the mean value coordinates.

Consider the unit sphere \( S = S(x, 1) \) centered at \( x \) and parametrized by the outer unit normal \( \nu \) (see Fig. 1(a) for a visual feedback on some notations used below). Let \( y \) be a point on \( \partial \Omega \) and \( \rho = |x - y| \). Denote by \( z \) the intersection point between the ray \( [xy] \) and \( S \). Assume that we know the values of function \( f(\cdot) \) on \( \partial \Omega \) and at \( x \). Then we can estimate \( f(z) \) using the linear interpolation:

\[
f(z) \approx \frac{(\rho - 1)f(x) + f(y)}{\rho}
\]

Now let us apply \( S \)-averaging to the left and right sides of the above equation:

\[
\int_S f(z) \, d\nu = |S| f(x) - f(x) \int_S \frac{d\nu}{|x - y|} + \int_S \frac{f(y) \, d\nu}{|x - y|}
\]

and, assuming that \( f(\cdot) \) is harmonic, we arrive at the transfinite mean value interpolation scheme of [JSW05b]

\[
f(x) = \int_S \frac{f(y) \, d\nu}{|x - y|} \bigg/ \int_S \frac{d\nu}{|x - y|}.
\]  (5)

This interpolation construction has a natural generalization

\[
f(x) = \int_S \frac{f(y) w(x, \nu) \, d\nu}{|x - y|} \bigg/ \int_S \frac{w(x, \nu) \, d\nu}{|x - y|},
\]  (6)

where \( w(x, \nu) \) is a weighting function associated with interpolated location \( x \).

Obviously (6) is just another form of (4) whose kernel satisfies the partition of unity property (2). One can also consider (6) as a transfinite version of the basic Shepard interpolation [She68].

Let us check whether (6) satisfies the linear precision property. For \( f(x) \equiv x \) we have \( f(y) \equiv y = x + \rho \nu \). Substituting the latter into (6) gives

\[
0 = \int_S \nu w(x, \nu) \, d\nu \text{ for each } x \in \Omega
\]  (7)

which is necessary and sufficient for linear precision.

One can see that (7) is satisfied if the weighting function is centrally-symmetric, i.e., coincides at each pair of antipodal points:

\[
w(x, \nu) = w(x, -\nu).
\]  (8)

Consider an important case of the planar interpolation: \( n = 2, \nu = (\cos \theta, \sin \theta) \),

\[
f(x) = \int_0^{2\pi} \frac{f(y) w(x, \theta) \, d\theta}{|x - y|} \bigg/ \int_0^{2\pi} \frac{w(x, \theta) \, d\theta}{|x - y|}.
\]  (9)

The orthogonality conditions

\[
\int_0^{2\pi} w(x, \theta) \cos \theta \, d\theta = 0, \quad \int_0^{2\pi} w(x, \theta) \sin \theta \, d\theta
\]  (10)

for each \( x \in \Omega \) are necessary and sufficient for linear precision.

It turns out that any barycentric interpolation scheme can be represented in the form of (6) with the weight function (which may be a measure of a generalized function) satisfying orthogonality conditions (7). Indeed, according to the Riesz representation theorem, given \( x \in \Omega \), a linear transfinite interpolation scheme can be considered as a linear functional \( T_x \) defined on an appropriate space of functions on \( \partial \Omega \)

\[
T_x : f|_{\partial \Omega} \rightarrow f(x)
\]

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and, therefore, can be represented by integration against a certain density $\mu_x$

$$f(x) \equiv T_x(f) = \int_{\partial \Omega} f(y) \mu_x(y) \, ds_y. \quad (11)$$

Denote by $h(x, y)$ the distance from $x$ to the supporting plane at $y \in \partial \Omega$, as seen in Fig. 1. The area (length) element $ds_y$ of $\partial \Omega$ at $y \in \partial \Omega$ can be expressed in polar coordinates $(\rho, \nu)$ centered at $x \in \Omega$ as

$$ds_y = \rho^n \, d\nu / h.$$  

Therefore we can rewrite (11) as follows

$$T_x(f) = \int_S f(y) \mu_x(y) \frac{|x-y|^n}{h(x, y)} \, d\nu. \quad (12)$$

It remains to set $w(x, \nu) = \mu_x(y)|x-y|^{n+1}/h(x, y)$.

Below we consider two important examples.

**Transfinite Laplace coordinates.** The transfinite Laplace coordinates are a continuous version of the discrete harmonic coordinates which are used widely in computational mechanics [SM06]. The discrete Laplace interpolation scheme was probably first time proposed in [CFL82] and then reinvented many times in connection with research on finite element methods [BVK*97] (the so-called non-Sibsonian interpolation), computational geometry [Sug99], and discrete minimal surfaces [PP99] (the so-called cotangent weights). A particular case of 2D transfinite Laplace coordinates was studied in [HS00] where interpolation of data defined on straight line segments was considered.

Following [SJW06] one can define the transfinite Laplace coordinates in a way similar to the derivation of the transfinite mean value coordinates if, instead of satisfying the mean value property, we search for the optimal value of $f(x)$ is given by

$$f(x) = \int_0^{2\pi} \left\{ f(y) - f(y) \left[ \frac{\rho_y}{\rho} \right]^2 + \left[ f(y) - f(x) \right] \frac{1}{\rho} \left[ \frac{1}{\rho} f_\theta(y) - \frac{\rho_y}{\rho} \right] \right\} \, d\theta$$

where the last integral is a quadratic function w.r.t. $f(x)$. Thus the optimal value of $f(x)$ is given by

$$f(x) = \int_0^{2\pi} \left\{ f(y) - f(y) \left[ \frac{\rho_y}{\rho} \right]^2 + \left[ f(y) - f(x) \right] \frac{1}{\rho} \left[ \frac{1}{\rho} f_\theta(y) - \frac{\rho_y}{\rho} \right] \right\} \, d\theta$$

where integrations by parts are used to derive (13).

Notice that (13) corresponds to (9) with $w(x, \theta) = \rho_\theta + \rho$.

A simple integration by parts shows that (14) satisfies the orthogonality conditions (10).

Probably the simplest way to define the transfinite Laplace coordinates in $\mathbb{R}^n$, $n \geq 3$, consists of expressing the coordinates via the gradient of area (volume), see [MBLD02, DMA02], for the relation between the discrete harmonic coordinates and the gradient of an area functional. Let $n(y)$ be the unit outer normal of $\partial \Omega$ at $y \in \partial \Omega$. Denote by $V(x)$ the volume of $\Omega$ as a function of $x$. Then, similar to the discrete case, we have

$$0 = \nabla_x V(x) = \int_{\partial \Omega} n(y) \, ds_y = \int_{\partial \Omega} \frac{n \, dn}{K(n)},$$

where $K(n)$ is the Gaussian curvature of $\partial \Omega$ parameterized by its unit normal. Thus, assuming that $\partial \Omega$ is parameterized by its Gauss map, the transfinite Laplace coordinates are given by (6) with

$$w(x, \nu) = 1 / K(\nu),$$

where $K(\nu)$ is the Gaussian curvature of $\partial \Omega$ at the point with normal $\nu$.

In 2D, (15) is obviously reduced to (14) (see also Section 6).

**Transfinite Wachspress-Warren coordinates.** Motivated by FEM applications, Wachspress [Wac75] proposed a construction of affine-invariant barycentric coordinates for convex polygons in 2D. Recently Warren and co-workers [War96, WSHD06] extended the Wachspress coordinates to convex polyhedra in $\mathbb{R}^n$ and presented a transfinite version of the coordinates.

Let us denote by $n(y)$ the outer unit normal to $\Omega$ at $y \in \partial \Omega$. The transfinite Wachspress-Warren coordinates are given by [WSHD06]

$$f(x) = \int_{\Omega} f(y) K(y) \rho(x, y) / h(x, y)^n \, ds_y / \int_{\Omega} K(y) \rho(x, y) / h(x, y)^n \, ds_y,$$
where \( K(y) \) is the Gaussian curvature of \( \partial \Omega \) at \( y \) and, as before, \( h(x, y) \) is the distance from \( x \) to the supporting plane at \( y \in \partial \Omega \).

Thus, in view of (12), the transfinite Wachspress-Warren coordinates are given by (6) with
\[
w(x, \nu) = K(y) \left[ \rho(x, y) / h(x, y) \right]^{n+1}. \tag{16}
\]

In 2D, straightforward computations show that (16) can be written as
\[
w(x, \theta) = (1/\rho)_y \theta + 1/\rho. \tag{17}
\]
A geometric proof of (17) is given in Section 6. Similar to (14), orthogonality conditions (10) for (17) are easily derived via integration by parts. Notice a similarity between (17) and (14). It will be explained in Section 6.

3. Pseudo-harmonic interpolation

Let us call an transfinite interpolation scheme pseudo-harmonic if it reproduces the harmonic functions in a ball.

Surprisingly, transfinite mean value interpolation (5) does not approximate harmonic functions. Indeed, let us consider the following pseudo-harmonic interpolation scheme
\[
f(x) = \int_{\partial \Omega} \frac{f(y) ds_y}{|x-y|^n} / \int_{\partial \Omega} \frac{ds_y}{|x-y|^n} \tag{18}
\]
which was introduced for \( d = 2 \) in [CMS98] as a transfinite version of a variant of the well known Shepard interpolation method [She68]
\[
f(x) = \frac{\sum_{i=1}^{N} f(y_i) |x-y_i|^{-n}}{\sum_{i=1}^{N} |x-y_i|^{-n}}.
\]
It is easy to see that (18) coincides with the Poisson Integral Formula for the Laplace equation when \( \Omega \) is a ball.

According to (12) we can rewrite (18) as follows
\[
f(x) = \int_S \frac{f(y) d\nu}{h(x, y)} / \int_S \frac{d\nu}{h(x, y)} \tag{19}
\]
which is another, quite elegant, form of the Poisson Integral Formula. \(^\dagger\)

In general, (19) (or, equivalently, (18)) does not give linear precision.

It is interesting to observe a similarity and difference between (19) and transfinite mean value coordinates (5). It also explains why (5) does not enjoy the property of being pseudo-harmonic.

Simple computations show that already in 2D the transfinite Laplace coordinates are not pseudo-harmonic. This is not surprising since in general the discrete harmonic coordinates lead to only zero-order consistency of the corresponding finite difference approximation of the Laplacian [Suk03]. However the discrete harmonic coordinates deliver a good approximation of the Laplace operator in Sobolev spaces of negative order [War05].

As shown in [FHK06], the discrete harmonic and Wachspress coordinates are the same for a circumscribable polygon. This implies that their transfinite versions coincide for a 2D circle. Thus the transfinite Wachspress-Warren coordinates are not pseudo-harmonic as well.

4. Weighted Gordon-Wixom coordinates

For 2D convex shapes, a simple and elegant construction for transfinite pseudo-harmonic barycentric coordinates was proposed by Gordon and Wixom in [GW74]. Below we describe their multidimensional analog.

Given a point \( x \) inside \( \Omega \), consider the unit sphere \( S \) centered at \( x \) and a unit normal \( \nu \in S \). Let the straight line through \( x \) determined by \( \nu \) intersect the boundary \( \partial \Omega \) in two points \( y_1 \) and \( y_2 \). Denote by \( \rho_1 \) and \( \rho_2 \) the distances from \( x \) to \( y_1 \) and \( y_2 \), respectively. Now we estimate \( f(x) \) using the linear interpolation between \( f(y_1) \) and \( f(y_2) \)
\[
f(x, \theta) = \frac{\rho_2}{\rho_1 + \rho_2} f(y_1) + \frac{\rho_1}{\rho_1 + \rho_2} f(y_2) = \frac{\left[ f(y_1) + f(y_2) \right] / \left[ 1 + \frac{\rho_1}{\rho_2} \right]}{1 + \frac{\rho_1}{\rho_2}} \tag{20}
\]
\( S \)-averaging w.r.t. \( \nu \) defines the Gordon-Wixom interpolation
\[
f(x) = \frac{1}{|S|} \int_S \left[ \frac{\rho_2}{\rho_1 + \rho_2} f(y_1) + \frac{\rho_1}{\rho_1 + \rho_2} f(y_2) \right] d\nu. \tag{21}
\]
As shown below for a more general situation, the Gordon-Wixom interpolation scheme is pseudo-harmonic and has linear precision.

Similar to weighted transfinite mean value coordinates (6), let us introduce a weighted version of the Gordon-Wixom interpolation scheme
\[
f(x) = \frac{\int_S \left[ \left( \frac{f(y_1)}{\rho_1} + \frac{f(y_2)}{\rho_2} \right) / \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) \right] W(x, \nu) d\nu}{\int_S W(x, \nu) d\nu} \tag{22}
\]
where \( W(x, \nu) \) is centrally-symmetric (8).
The Intersecting Chords Theorem is usually formulated for a 2D circle. However its extension onto the multidimensional case is straightforward.

Observe that if weighting function of (6) satisfies (8) (centrally-symmetric), then the weighting functions of (22) and (6) are related to each other by

\[ W(x, \nu) = \frac{w(x, \nu)}{\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)}. \]

In particular, (22) with \( W(x, \nu) = 1/\rho_1 + 1/\rho_2 \) is reduced to mean value coordinates (5).

Now let us consider a particular case of (22) obtained when \( W(x, \nu) \) depends on distances \( \rho_1 \) and \( \rho_2 \) only. Since \( W(x, \nu) \) is centrally-symmetric, it can be written as a function of \( \rho_1 \cdot \rho_2 \) and \( \rho_1 + \rho_2 \)

\[ W(x, \nu) = W(\rho_1 + \rho_2, \rho_1, \rho_2). \]  

Let us show that if (23) depends only on the product of \( \rho_1 \) and \( \rho_1 \), then barycentric coordinates (22) are pseudo-harmonic. Indeed if \( \Omega \) is a ball then according to the Intersecting Chords Theorem

\[ \rho_1 \rho_2 \equiv c(x), \]

where \( c(x) \) depends only on \( x \). Thus it is sufficient to demonstrate that (21) is pseudo-harmonic. Notice that we can exploit symmetry properties of (21) and rewrite it as

\[ f(x) = \frac{2}{|S|} \int_{S/2} \frac{\rho_2}{\rho_1 + \rho_2} f(y_i) \, d\nu, \]  

where by \( S/2 \) we denote a unit semisphere. Let \( \Omega \) be a ball of radius \( R \). Consider the 2D disc obtained as the intersection between \( \Omega \) and the plane formed by \( y_1, y_2 \), and the center of the ball. In notations of Fig. 1(b), we obviously have

\[ h_1 = \rho_1 \sin \varphi, \quad h_2 = \rho_2 \sin \varphi, \quad \rho_1 + \rho_2 = 2R \sin \varphi, \quad h_1 h_2 = c(x), \quad \frac{\rho_2}{\rho_1 + \rho_2} = \frac{h_2}{2R \sin \varphi} = \frac{c(x)}{2R}. \]

Thus (24) is reduced to (19). A slightly more complex proof of this result for the 2D version of (21) is given in [GW74] and attributed to W. W. Meyer.

Now we can see why the original Gordon-Wixom interpolation scheme is good in approximating harmonic functions. The second-order directional derivative of \( f(x) \) in the \( \nu \)-direction can be approximated by

\[ f''(x) \approx D_{\nu\nu}[f(x)] = \frac{2}{\rho_1 \rho_2} \left( \frac{f(y_1)}{\rho_1} + \frac{f(y_2)}{\rho_2} \right) \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) - 2f(x). \]

Since

\[ 2\Delta f(x) = \frac{1}{|S|} \int_S f''(x) \, d\nu, \]

we obtain weighted Gordon-Wixom interpolation (22) with weight

\[ W(x, \nu) = \frac{1}{\rho_1 \rho_2} \]  

and, according to the Intersecting Chords Theorem, arrive at (21) when \( \Omega \) is a ball.

As mentioned before, weighting functions \( w(x, \nu) \) in (6) and \( W(x, \nu) \) in (22) may be generalized functions. In particular, Gordon and Wixom [GW74] considered a transfinite interpolation scheme which can be obtained from (22) if we set \( W(x, \nu) = \delta(\nu - \nu_0) \), where \( \delta(\cdot) \) is the Dirac delta function and \( \nu_0 \) is a given direction.

Non-convex domains. So far we have assumed that \( \Omega \) is convex. It turns out that it is very easy to extend the weighted Gordon-Wixom interpolation scheme (22) to generic non-convex domains. Consider the straight line \( l(x, \nu) \) through \( x \) at direction \( \nu \) intersecting \( \partial \Omega \) in \( m \) points \( y_1, \ldots, y_m \). Let us set \( \epsilon_i = 1 \) if the ray \( [x, y_i] \) arrives at \( y_i \) from inside of \( \Omega \), \( \epsilon_i = -1 \) if the ray approaches \( y_i \) from outside of \( \Omega \), and \( \epsilon_i = 0 \) if the ray is tangent to \( \partial \Omega \) at \( y_i \). Define

\[ f(x, \nu) = \sum_{i=1}^{m} f(y_i) w(x, y_i) \frac{\epsilon_i}{\rho_i} / \sum_{i=1}^{m} w(x, y_i) \frac{\epsilon_i}{\rho_i}, \]  

where \( \rho_i = |x - y_i| \). This simple one-dimensional construction belongs to the family of barycentric rational interpolation schemes studied widely in constructive approximation [BBM05]. Now \( \nu \)-averaging w.r.t. \( \nu \)

\[ f(x) = \frac{1}{|S|} \int_S f(x, \nu) \, d\nu \]  

gives an extension of (22) to generic non-convex domains.

If \( \Omega \) is not generic, it may happen that \( l(x, \nu) \cap \partial \Omega \) contains linear segments. It is natural to treat such a situation as the tangency case and set \( \epsilon = 0 \) in (27) for all the points of those linear segments on \( l(x, \nu) \).

Instead of (27) one can use high-order 1D barycentric interpolation schemes introduced very recently by Floater and Hormann [FH06].

An example of interpolating inside of a non-convex polygon with (27) is shown in Fig. 1(c). Similar to [GW74] we evaluate (28) numerically.

Gordon-Wixom coordinates for polygons. Let \( \Omega \) be a 2D convex polygon and \( f(y), y \in \partial \Omega \), be given by its values at the polygon vertices. Assume that \( f(y) \) is linear on the edges of the polygon. Then (21) allows
for a closed-form solution. For a planar domain, one can rewrite (21) as
\[ \frac{1}{\pi} \int_{0}^{2\pi} \frac{\rho_2}{\rho_1 + \rho_2} f(y_1) \, d\theta. \] (29)

Consider point \( x \) inside polygon \( \Omega \) and place new vertices on the boundary of the polygon such that the straight line connecting \( x \) with an original vertex intersects \( \partial \Omega \) in another, possibly new, vertex. See Fig. 1(d) where \( A \) and \( D \) are original polygon vertices and \( B \) and \( C \) are new vertices. Then we repeat trigonometric calculations of [Flo03] and, in the notations of Fig. 1(d), arrive at
\[ \rho_1(\theta) = \frac{ab \sin \alpha}{a \sin \theta + b \sin(\alpha - \theta)}, \]
\[ \lambda_1 = \frac{b \sin(\alpha - \theta)}{a \sin \theta + b \sin(\alpha - \theta)}, \]
\[ \lambda_2 = \frac{a \sin(\theta)}{a \sin \theta + b \sin(\alpha - \theta)}, \]
where \( \lambda_1(\theta) \) and \( \lambda_2(\theta) \), \( \lambda_1 + \lambda_2 = 1 \), are the weights for the linear interpolation inside the segment AB. We compute \( \rho_2(\theta) \) in a similar way and get the following formula for the part of (29) corresponding to AB.
\[ \frac{1}{\pi} \int_{0}^{\alpha} \frac{cd}{ac(b + d) \sin \theta + bd(a + c) \sin(\alpha - \theta)} \, d\theta. \] (30)

Although (30) looks quite complex, it can be evaluated in closed form (we have used Maple to express (30) in terms of elementary functions). Unfortunately the result is too lengthy to present here.

5. Barycentric coordinates and PDEs

**Quasi-Laplacian.** Let us consider the Dirichlet boundary value problem for a quasi-Laplacian operator:
\[ \nabla \cdot (a(x) \nabla f(x)) = 0 \quad \text{in} \quad \Omega, \quad f(x) \text{ is known on } \partial \Omega, \] (31)
where \( a(x) > 0 \) is a known conductivity coefficient. Integration by parts gives
\[ \int_{S(x, r)} \frac{\partial f}{\partial \nu} \, d\nu = 0, \] (32)
where, as before, \( S(x, r) \) is the sphere of radius \( r \) centered at \( x \) and contained inside \( \Omega \) and \( \nu \) is outward unit normal to \( S(x, r) \). Let us group opposite points of \( S_+ \) together in (32):
\[ a \frac{\partial f}{\partial \nu} \bigg|_{x + r(\nu)} - a \frac{\partial f}{\partial \nu} \bigg|_{x - r(\nu)} \approx 2r \frac{\partial}{\partial \nu} \left( a(x) \frac{\partial f(x)}{\partial \nu} \right) = 2r \left[ a'_\nu(x) f'_\nu + a(x) f''_\nu \right]. \] (33)

We estimate the first- and second-order directional derivatives of \( f(\cdot) \) by
\[ f'_\nu(x) \approx \frac{f(y_1) - f(x)}{\rho_1}, \]
and (25), respectively. Substituting these approximations into the right-hand-side of (33), integrating over a half-sphere \( S(x, r)/2 \), and taking into account (32) yields
\[ 0 \approx \int_{S/2} \frac{f(y_1) - f(x)}{\rho_1} a'_\nu(x) \, d\nu + a(x) \int_{S/2} D_{\nu\nu}[f(x)] \, d\nu \] (34)
with \( D_{\nu\nu}[f(x)] \) defined in (25). We rewrite the integrals in the right-hand side of (34) as integrals over the whole unit sphere \( S \) and arrive at
\[ f(x) \int_{S} a'_\nu(x) \, d\nu + a(x) \int_{S} D_{\nu\nu}[f(x)] \, d\nu \]
which can be considered as a combination of the (26)-weighted Gordon-Wixom interpolation scheme and \( a'_\nu(x) \)-weighted transfinite mean value coordinates.

Note that it does not satisfy the linear precision property because of weighting function \( a'_\nu(x) \) in the mean value component. Notice however that linear functions do not satisfy (31) unless the conductivity coefficient \( a(x) \) is constant.

Of course, linear PDE (31) is rather a toy example. However a similar approximation approach can be applied to quasi-linear PDE operators in the form
\[ \frac{2}{|S|} \int_{S/2} \partial_\nu \left( g \left( |\partial_\nu f| \right) \partial_\nu f \right) \, d\nu, \] (35)
where \( g(\cdot) \) is a positive function. A 2D version of (35) was used in [Wei94] for anisotropic nonlinear image diffusion purposes.

**AMLE.** According to an axiomatic approach to image interpolation developed in [CMS98] and numerical experiments conducted in [ACGR02], absolutely minimizing functions [ACJ04] satisfy many properties desirable for a feature-preserving interpolation of height data.

Weighted Gordon-Wixom coordinates (21) approximate the solution to the AMLE equation if we set \( W(x, \nu) = \delta(\nu - \nu_0(x)) \), where \( \nu_0(x) \) is a direction for which the right-hand side of
\[ f'_\nu(x) \approx \frac{f(y_1) - f(y_2)}{\rho_1 + \rho_2}, \] (36)
attains its maximal absolute value. Similar to the AMLE interpolation, (36) is capable of interpolating isolated values. For example, let \( \Omega \) be the unit disk without its center (a punctured disk). Define \( f(y) = 1 \) at the disk center and \( f(y) \equiv 0 \) on the outer boundary of the punctured disk. Then the solution to (36) is given by cone \( f(x) = 1 - |x| \). Exactly the same result is delivered by AMLE.
6. Christoffel-Minkowski type problems and barycentric coordinates

So far we have studied abilities of various barycentric interpolation schemes to approximate solutions to second-order elliptic PDEs. In this section, we reveal and discuss surprising links between the barycentric coordinates and classical inverse problems of differential and convex geometry.

Minkowski problem. The classical Minkowski problem is an inverse problem in differential geometry and concerns reconstruction of a closed convex hypersurface from its Gaussian curvature given as a function of the outer surface normal [Min03]. Given a positive function \( K(\nu) \) defined over the unit sphere \( S \), a necessary and sufficient condition of the Minkowski problem is

\[
\int_{S} \nu \, dl = \int_{\Sigma} \frac{\nu \, d\nu}{K(\nu)} = 0,
\]

where \( \Sigma \) is the reconstructed hypersurface, \( dl \) its area element and \( d\nu = K \, d\nu \) by definition of the Gaussian curvature. The necessity of (37) follows immediately from the divergence theorem of vector calculus. The sufficiency is non-trivial and was proven, under various assumptions of smoothness of \( \Sigma \), by Minkowski himself (1903), Alexandrov (1938), Lewy (1938), Miranda (1939), Pogorelov (1952), Nirenberg (1953), Cheng and Yau (1976), and others.

Exploring the similarity between (37) and (7), we get for free the following result delivering a geometric description of barycentric interpolation schemes. For each \( x \in \Omega \), consider a family of hyperplanes in \( \mathbb{R}^n \)

\[
(z - x) \cdot \nu = p(x, \nu)
\]

parameterized by \( \nu \in S \). For a given \( \nu \), \( p(x, \nu) \) stands for the signed distance from the plane defined by \( \nu \) to \( x \). We assume that \( p(x, \nu) \) is sufficiently smooth. Denote by \( \Sigma_x \) the envelope of family (38) and let \( K_x(\nu) \) be the Gaussian curvature of \( \Sigma_x \). Assuming that \( \Sigma_x \) is convex (and, therefore, its Gaussian curvature is positive), we obtain transfinite barycentric coordinates (6) with \( w(x, \nu) = 1/K_x(\nu) \). Vice versa, given barycentric coordinates (6) whose weighting function \( w(x, \nu) \) is positive at \( x \), a convex hypersurface \( \Sigma_x \) is reconstructed from its Gaussian curvature

\[
K_x(\nu) = 1/w(x, \nu).
\]

In order to use the Minkowski problem for a geometric description of barycentric coordinates (6) whose weighting function \( w(x, \nu) \) is not always positive, one can consider the so-called hedgehogs, closed surfaces parameterized by their Gauss maps and described as the envelopes of their tangent planes. Unfortunately, if \( n \geq 3 \), (37) is not sufficient in the case of envelopes forming non-convex surfaces [MM01](Proposition 7).

Christoffel-Minkowski problem. A generalization of the Minkowski problem, the so-called Christoffel-Minkowski problem, consists of finding a convex hypersurface \( \Sigma \) with a prescribed elementary symmetric polynomial of the surface principal radii [GG02]. For \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{R}^{n-1} \), let \( S_k[\lambda] \) be an elementary symmetric polynomial of degree \( k \)

\[
S_k[\lambda] = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},
\]

where the sum is taken over all permutations of the indices \( \{1, \ldots, n-1\} \). Denote by \( R_i(\nu) = 1/k_i(\nu) \) the surface principal radii parameterized by the Gauss map of the surface. Direct computations (see, for example, [Bla30, BF34, Bus58, Sch93]) show that \( R_i(\nu), i = 1, \ldots, n-1 \), are the eigenvalues of the matrix

\[
\nabla^2_{\nu} p(\nu) + p(\nu) I,
\]

where \( \nabla^2_{\nu} \) is the Hessian operator w.r.t. a local orthonormal frame on \( S \) and \( I \) is the identity matrix. It can be shown [BF34, Bus58, Sch93] that

\[
\int_{S} \nu S_k[R_1, \ldots, R_{n-1}](\nu) \, d\nu = 0.
\]

The Christoffel-Minkowski problem consists of determining a convex surface whose curvature radii \( R_1(\nu), \ldots, R_{n-1}(\nu) \) satisfy

\[
S_k[R_1, \ldots, R_{n-1}](\nu) = \varphi(\nu),
\]

where \( \varphi(\nu) \) is a given function such that

\[
\int_{S} \nu \varphi(\nu) \, d\nu = 0.
\]

Orthogonality condition (41) generalizes (37) and can be used for a geometric characterization of transfinite barycentric coordinates. A substantial progress in solving the general Christoffel-Minkowski problem for convex bodies was recently achieved in [GG02, STW04].

The Monge-Ampère equation

\[
K(\nu) \equiv \det \left[ \nabla^2_{\nu} p(\nu) + p(\nu) I \right] = \varphi(\nu)
\]

used to solve the Minkowski problem is a particular case of (40) which arises when the product of the curvature radii is considered:

\[
S_{n-1}[R_1, \ldots, R_{n-1}] = R_1 \cdots R_{n-1} \equiv 1/K.
\]

Christoffel problem. Another particular case of the Christoffel-Minkowski problem is obtained from (40) when \( k = 1 \). It gives the so-called Weingarten formula [Wei84] (as cited in [Bla30])

\[
\text{trace} \left[ \nabla^2_{\nu} p(\nu) + p(\nu) I \right] \equiv \Delta_{\nu} p(\nu) + (n-1)p(\nu) = R(\nu),
\]

(42)
where $\Delta_\nu$ is the spherical Laplacian and

$$R(\nu) = \sum_{i=1}^{n-1} R_i(\nu)$$

is the sum of the principal curvature radii (the reciprocal of the sum is the so-called harmonic curvature). Since $\lambda = (n - 1)$ is the first eigenvalue of $\Delta_\nu$ and $\nu$ are the corresponding $n - 1$ eigenfunctions, the necessary and sufficient condition of solvability (42) is given by

$$\int_{\Sigma} \nu R(\nu) \, ds = 0.$$  \hfill (43)

In 3D, this particular case of the Christoffel-Minkowski problem was proposed and studied by Christoffel in 1865 [Chr65] and seems to be the earliest inverse problem in differential and convex geometry. For a relatively recent account, see, for example, [Sch93](Section 4.3),

Given barycentric coordinates (6) with weighting function $w(x, \nu)$ satisfying (7), for each $x \in \Omega$ we set

$$R(x, \nu) = w(x, \nu)$$

and solve linear second-order elliptic PDE (42) Then surface $\Sigma_x$ is reconstructed from support function $p(x, \nu)$. (The simplest way to solve (42) consists of expanding $w(x, \nu)$ into spherical harmonics and constructing support function $p(x, \nu)$ of $\Sigma_x$ as a spherical harmonic series.)

**Christoffel-Minkowski problems in 2D and 3D.**

In 2D, the Christoffel and Minkowski problems coincide and lead to the following second-order differential equation

$$p''(\theta) + p(\theta) = \frac{R(\theta)}{1/k(\theta)}$$  \hfill (44)

for the support function $p(\theta)$. Curve $\Sigma$ is then obtained from its support function $p(\theta)$ in the following parametric form

$$z(\theta) = \left[ \begin{array}{c} z_1 \\ z_2 \end{array} \right] = \left[ \begin{array}{c} x_1 \\ x_2 \end{array} \right] + \left[ \begin{array}{c} p(\theta) \cos \theta + p'(\theta) \sin \theta \\ p(\theta) \sin \theta + p'(\theta) \cos \theta \end{array} \right].$$  \hfill (45)

For our purposes, for each $x \in \Omega$ we determine $\Sigma_x$ by solving

$$p''(\theta) + p(\theta) = w(x, \theta)$$ \hfill (46)

and then using (45). If $w(x, \theta)$ is a positive function of $\theta$, then $\Sigma_x$ forms a convex curve. If $w(x, \theta)$ changes its sign, $\Sigma_x$ has cusps at the points corresponding to the zeros of the curvature radius $R(\theta) = w(x, \theta)$.

Fig. 2 shows an example of a planar curve generated from its support function.

In 3D, we have two types of the Christoffel-Minkowski problems corresponding to the Gaussian curvature (the Minkowski problem) and harmonic curvature (the Christoffel problem). For the Christoffel problem, $\Sigma_x$ is reconstructed by solving

$$\Delta_\nu p(\nu) + 2p(\nu) = w(x, \nu).$$  \hfill (47)

Then one can reconstruct $\Sigma_x$ from its support function $p(\nu)$ using formulas derived in [VF92]. Similar to the 2D case, $w(x, \nu)$ may change its sign and $\Sigma_x$ form cuspedal edges at the points where its harmonic curvature $1/w(x, \nu)$ becomes infinite.

![Figure 2: (a) Polar plot of a given support function $p(\theta)$. (b) Graph of the curvature radius $R(\theta) = p''(\theta) + p(\theta)$. (c) The curve reconstructed from its support function $p(\theta)$. The curve has six cusps corresponding to the zeros of its curvature radius $R(\theta)$.

**Pedal and negative pedal surfaces.**

Pedal curves and surfaces are differential geometry objects which have a special significance in classical geometric optics [Her00] and are often mentioned in textbooks on classical differential geometry. Given a surface $\Sigma$ and a point $x$, the pedal of $\Sigma$ w.r.t. $x$ is defined as the locus of the feet of the perpendiculars from $x$ to the tangent planes to the surface. The negative pedal curves and surfaces are much less mentioned in the differential geometry literature in spite of the fact that they can be used for designing mirrors with prescribed properties [FMR01]. The negative pedal of $\Sigma$ w.r.t. $x$ is constructed as the envelope of planes passing through the points of $\Sigma$ and perpendicular to the segments connecting $x$ with the points of $\Sigma$. The pedal and negative pedal curves and surfaces are also studied in the projective geometry framework [CG67](Chapter 6).

Pedals and negative pedals find applications in materials science [POMZ99]. Following [POMZ99] let us consider two Legendre transformations of positive functions defined on the unit sphere $S$. Let $\tau : S \to \mathbb{R}_+$ be a continuous function. The polar plot of $r(\nu), \nu \in S$, is the surface (curve in 2D) defined in the $\{r, \nu\}$ polar coordinates by $r = r(\nu)$. The first Legendre transform $r_+(\omega)$ is the function corresponding to the negative pedal of the polar plot of $r(\nu)$. The second Legendre transform $r''(\nu)$ is the function corresponding to the pedal of the polar plot of $r(\nu)$.

As demonstrated in [POMZ99], the following dual-
ity relations are hold
\[ 1/r^* = [1/r]_+ , \quad 1/\rho = [1/\rho]_+. \] (48)
In addition, if \( r(\nu) \) satisfies certain convexity conditions, then
\[ [r_+]^* (\nu) \equiv r(\nu) \equiv [r^*]_+(\nu). \] (49)

**Geometry of Wachspress-Warren coordinates.**

Now we are ready to establish a link between the transfinite Wachspress-Warren coordinates and the Minkowski problem. As before, given a convex bounded domain \( \Omega \) in \( \mathbb{R}^n \) and point \( x \in \Omega \), let \( r = \rho(\nu) \) be the polar plot of \( \partial \Omega \) w.r.t. \( x \). \( K(y) \) denote the Gaussian curvature of \( \partial \Omega \) at \( y \in \partial \Omega \), and \( h(x, y) = (y - x) \cdot n(y), \) where \( n(y) \) is outer unit normal to \( \partial \Omega \) at \( y \), be the support function of \( \partial \Omega \) w.r.t. \( x \). Consider surface \( \Sigma_x \) whose support function w.r.t. \( x \) is given by \( 1/\rho(\nu) \). Using duality relations (48) and (49) it is easy to show that the reciprocal of the Gaussian curvature of \( \Sigma_x \) is given by \( K_\nu^{n+1}/h^{n+1} \). Thus, according to (37), we have proven the linear precision property of transfinite Wachspress-Warren coordinates (6), (16). In the 2D case, (17) now follows from (44).

An equivalent interpretation of the transfinite Wachspress-Warren coordinates in terms of pedal surfaces (polar duals) was considered in [WSHD06, SJW06] (see also [JSW05a] where a similar description of the discrete Wachspress-Warren coordinates is presented).

**Geometry behind Laplace coordinates.**

According to (15), the geometry of the transfinite Laplace coordinates is simpler than that of Wachspress-Warren coordinates. It is reduced to the Minkowski problem for \( \partial \Omega \) (therefore, it can be called by the Minkowski extension of the 2D Laplace coordinates). For each \( x \in \Omega \), the support function of \( \Sigma_x \) is given by \( r = \rho(\nu) \), the polar plot of \( \partial \Omega \) w.r.t. \( x \). The linear precision is guaranteed by (37).

It seems, however, that the Christoffel extension of the 2D Laplace coordinates is more appropriate for approximating harmonic functions than the Minkowski extension considered in the previous paragraph. Indeed, let us assume that functions \( p(\nu) \) and \( R(\nu) \), \( \nu \in \mathbb{S} \), in (43) are defined in \( \mathbb{R}^n \) as homogeneous of degree one: \( p(tx) \equiv tp(z) \) and \( R(tx) \equiv tR(z) \), \( z \in \mathbb{R}^n \). Then (43) can be rewritten as
\[ \Delta s p(z) = R(z) \] (50)
and the left-hand side of (50) with \( p(z) = \rho(z) \), where \( \rho(\nu) \) is the polar plot of \( \partial \Omega \), defines the weighting function
\[ w(x, \nu) = R(z|z) \] (51)
corresponding to the Christoffel extension of the 2D Laplace coordinates. The Laplacian from the left-hand side of (50) indicates that (6) with (51) is a proper choice for approximating harmonic functions.

**Weighted Gordon-Wixom coordinates.**

As mentioned before, weighted Gordon-Wixom coordinates (22) are equivalent to (6) with centrally-symmetric weighting: \( w(x, \nu) = w(x, -\nu) \). Therefore, surfaces \( \Sigma_x \) constructed as solutions to a Christoffel-Minkowski problem are also centrally-symmetric. Centrally-symmetric convex bodies possess interesting properties and are widely studied in convex geometry [BF34, Sch93].

**Geometry behind mean value coordinates.**

Since for the mean value coordinates \( \Sigma_x \) is always the unit sphere \( \mathbb{S} \), these coordinates play a distinguish role in the above inverse problem constructions and correspond to the case when curvature function \( S_1[R_1, \ldots, R_{n-1}] (\nu) \) from (40) is constant.

**Algebra of barycentric coordinates.**

Given two sets defined by their support functions, say \( p(\nu) \) and \( q(\nu) \), \( \nu \in \mathbb{S} \), one can consider algebraic operations over the sets by adding (Minkowski addition), multiplying, and convolving the support functions.

As observed before, the support functions of surfaces \( \Sigma_x \) corresponding to the Laplacian and Wachspress-Warren coordinates are given by \( \rho(\nu) \) and \( 1/\rho(\nu) \), respectively, where \( \rho(\nu) \) is the polar plot of \( \partial \Omega \). Thus their product gives us the support function corresponding to the mean value coordinates and we can say that the Laplacian and Wachspress-Warren coordinates are dual w.r.t. to the multiplication of their corresponding support functions.

As noted in [MM06](Section 6, see also references to works of H. Görtler therein), the convolution of two support functions inherits properties of the factors. For the sake of simplicity, let us consider the 2D case. The convolution of \( p(\theta) \) and \( q(\theta) \) is given by
\[ (p \otimes q)(\theta) = \frac{1}{2\pi} \int_0^{2\pi} p(\theta - \alpha) q(\alpha) d\alpha. \] (52)
If \( p(\theta) \) is centrally-symmetric, i.e., \( p(\theta + \pi) \equiv p(\theta) \), so is convolution \( p \otimes q \). If \( q(\theta) \) is of constant-width (Fig. 2 gives an example of a constant-width curve generated from its support function),
\[ q(\theta + \pi) + q(\theta) \equiv \text{const}, \]
then \( p \otimes q \) is also of constant width. Notice that a circle is the only centrally-symmetric set of constant width.

According to (46), if support function \( p(\theta) \) of \( \Sigma_x \) is centrally-symmetric/constant-width, so is weighting function \( w(z, \theta) \), and vice versa. Since the support functions corresponding to the weighted Gordon-Wixom coordinates are centrally-symmetric, then
“constant-width coordinates” are dual to the weighted Gordon-Wixom ones w.r.t. convolution (52).

7. Directions for further work
As shown in [SJW06], transfinite barycentric interpolation schemes can be viewed as limiting cases of their corresponding discrete versions, barycentric coordinates on polyhedral domains. Certainly the discrete case is more complex than the continuous one considered in this paper. For example, several different discrete counterparts of the transfinite barycentric coordinates are studied in [FHK06, JW05]. Establishing links between discrete barycentric coordinates and discrete Christoffel-Minkowski problems constitutes an interesting topic for further work.

In this paper, we haven’t even touched on the transfinite Sibson coordinates [GF99] and Möbius-invariant interpolation [BE03]. Following [GW74], it would be also interesting to consider the Hermite data interpolation problem and study its relationships with fourth-order PDEs.

Another fascinating theme for future research consists of studying links between transfinite barycentric coordinates and continuous valuations on convex sets. [Had57, Sch93]. Some links between the Hadwiger characterization theorem of rigid motion invariant valuations [Had57] and barycentric coordinates are pointed out in [DMA02]. An interesting problem here is to study relations between barycentric coordinates and a more general class of translation invariant valuations [Sch96].

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References


Alexander Belyaev / On Transfinite Barycentric Coordinates


