On the Fractal Behaviour of AA Patterns

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Abstract
"AA Patterns" is a recently discovered kind of algorithmic art in the form of pixel patterns; where each pixel in a 2D bitmap is set or unset according to a simple test applied to its coordinates pair. In spite of their iteration-free algorithm, AA Patterns exhibit signs which suggest some relation to fractals. This paper investigates this relationship, and reveals a new fractal which comes from an iteration-free process.

Categories and Subject Descriptors (according to ACM CCS): I.3.0 [Computer Graphics]: General—

1. Introduction

"AA Patterns" refer to a recently developed kind of algorithmic art, in which a simple affine transformation is employed to generate pixel patterns exhibiting some symmetries [Ahm11c], like those shown in Figure 1. For every distinct real number α between 1 and 2, an AA Pattern, $AA(\alpha)$, is defined as the set of integer points (X,Y) which do not satisfy the equation

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \left\lfloor \frac{1}{2} \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\rfloor \quad 1 < \alpha < 2 \quad (1)$$

for any integer pair (x,y). Different algorithms were developed in [Ahm11c] to generate and color AA Patterns, and many of their properties were discussed in [Ahm11a] and [Ahm11b].

As apparent in Figure 1, the constituent points of AA Patterns group in 'clusters' which have gradually increas-
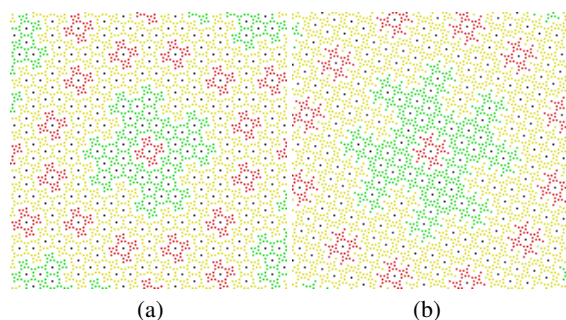

(a)   (b)

Figure 1: *Example AA Patterns.*

ing levels of complexity, and they exhibit a form of self-similarity between these cluster levels. Self-similarity is a cornerstone of fractals (see [Man83]), which suggests some relationship between AA Patterns and fractals. On the other hand, AA Patterns miss another common characteristic of fractals, which is iteration. Indeed, generating AA Patterns is an iteration-free process which can be optimized for extremely fast execution. Thus, if the exact relationship between AA Patterns and fractals is revealed, then that might help in developing iteration-free fractal generating algorithms controlled by real parameters.

2. Fractal properties of AA Patterns

According to Falconer [Ken03], a typical fractal set (i) has fine structure (detail on arbitrarily small scales), (ii) is too irregular to be described in traditional geometrical language, both locally and globally, (iii) has some form of self-similarity, perhaps approximate or statistical, (iv) has a 'fractal dimension' greater than its topological dimension, and (v) is defined in a very simple way, perhaps recursively.

In the light of these properties of fractals let us highlight some properties of AA Patterns. In doing so we are by no means trying to prove that AA Patterns are fractals; rather, we are trying to show signs of a possible relationship between the two. Indeed, AA Patterns are defined in a simple way, yet they are too irregular to be described in traditional geometrical language.

The continued fraction expansion (CF) of the parameter

$$\alpha = [1; m, \alpha_1, \alpha_2, \alpha_3, \ldots] \quad (2)$$

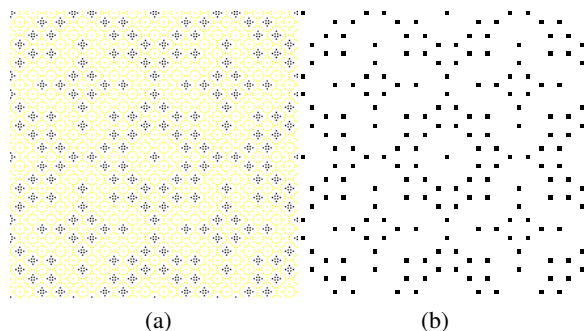plays an analogous role to iteration in fractals. It was demon-

Figure 2: *Level-2 clusters of (a) AA([1; 1, 2, 2, 3, 2]) are distributed in a similar manner to points in (b) AA([1; 2, 3, 2]). Notice the partial similarity in continued fractions.*

Figure 3: *Scaling self-similarity of AA $\left(\sqrt{3}\right)$ illustrated by zooming at (a) level-3 and (b) level-4. Notice how each cluster in (a) is replaced by a higher-level cluster in (b). Colors of the second plot were shifted so as to highlight similarity. Plot (a) is 112×112, and plot (b) is 418×418; and the scale between the plots $\approx 2 + \sqrt{3}$.*

Figure 4: *Color maps used to color (a) level-3 and (b) level-4 approximations of AA($\sqrt{3}$) shown in Figure 3. Besides the apparent similarity between the two maps, map (a) is identical to 12 regions in map (b).*

strated in [Ahm11b] that each one or two additional entries α_i add new clusters in $AA(\alpha)$ at a higher level of complexity; the same way that each additional iteration adds detail to, say, Koch's curve. Another way to state it is that truncating the CF at a certain entry limits the pattern to the corresponding level of detail; and the truncated pattern resembles the original up to that level.

Proceeding with our analogy, we can think of patterns with irrational parameters, infinite entries in CF, and infinite levels of detail, as the original AA Patterns; and ones with rational parameters (k entries in CF) as approximations only; the same way that for a fractal set F, "'pictures of F' tend to be pictures of one of the E_k, which are a good approximation to F when k is reasonably large" [Ken03]. Note, however, that whereas details in fractals usually appear at smaller scales, details in AA Patterns appear at larger scales, as we zoom out.

Even though AA Patterns are not defined recursively, they posses recursive structures. In Figure 1, for example, each cluster looks as if it was recursively synthesized from parts of smaller clusters; and each cluster recursively houses, and is recursively surrounded by, a set of smaller clusters.

AA Patterns are self-similar in the sense that the set of clusters at each level of complexity is distributed in a (visually) similar form to an AA Pattern; as illustrated in Figure 2. For some parameters with periodic CF (see [Ahm11b]) the distribution of clusters resembles the pattern itself, manifesting self-similarity, as illustrated in Figure 3. We might think of these as 'scaling' AA Patterns.

We should not leave this section without having a look at the color maps used to color AA Patterns, like the two shown in Figure 4. More about these color maps and how they are constructed and used can be found in [Ahm11c]. What is relevant here is their fractal-like shape. At first glance they might give the impression of a variant of Sierpinski carpet, where instead of removing squares in each iteration they are just painted in a different color.
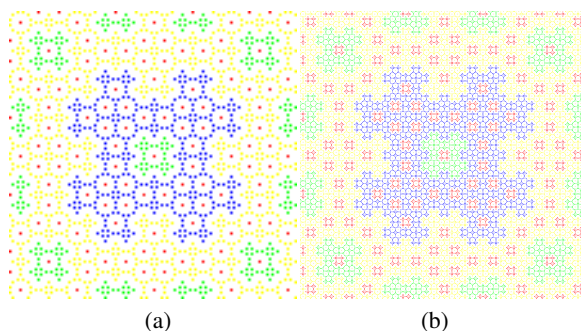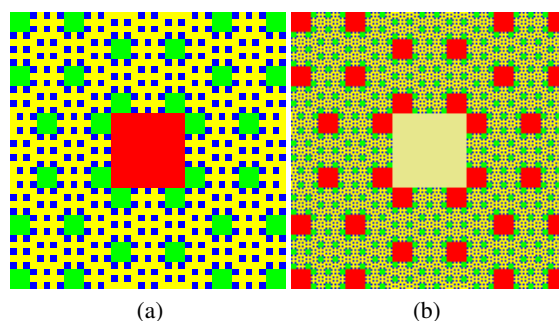
3. Analysis of algorithms

Having seen some fractal symptoms in AA Patterns, we will try to analyze the algorithms that plot them to see how these could generate a fractal, if any. As pointed out in [Ahm11a], the value of m in (2) affects only the dispersion of points, but not the general shape of the pattern, so we can safely focus our analysis on the case of

$$m = 1. \qquad (3)$$

Under this constraint we revoke two important sub-parameters from [Ahm11c]:

$$t = \frac{\alpha}{2} - \frac{1}{2}, \qquad (4)$$

$$r = \frac{1}{2} - t = 1 - \frac{\alpha}{2}, \qquad (5)$$

$$r < \frac{1}{4} < t < \frac{1}{2}. \qquad (6)$$

The original plotting algorithm [Ahm11c], which follows directly from (1), plots the complement set of AA Patterns,

Algorithm 1 Plotting a transformed version of AA Patterns. Adapted from [Ahm11c].

```
For all integer points (x, y)
  dX = frac((alpha*x − y)/2)
  dY = frac((alpha*y + x)/2)
  if (dX < t) and (dy < t)
    set (x, y)
```

Algorithm 2 Splitting Algorithm 1 into two passes.

```
For all integer values of y
  For all integer values of x
    dX = frac((alpha*x − y)/2)
    if (dX < t)
      set (x, y) in plot1
For all integer values of x
  For all integer values of y
    dY = frac((alpha*y + x)/2)
    if (dY < t)
      set (x, y) in plot2
For all integer points (x, y)
  if (x, y) is set in plot1 and plot2
    set (x, y) in final plot
```

which hardly helps here. We skip forward and look at Algorithm 1, which was developed in [Ahm11c] to plot a transformed version of AA Patterns. The function "frac" extracts the fractional part of its argument:

$$\mathrm{frac}(x) = x - \lfloor x \rfloor \tag{7}$$

We start our inspection by splitting the joint condition into two separate conditions, effectively turning the plotting process into a two-pass setup, as shown in Algorithm 2. The first-pass plot can be decoupled further by distributing the fraction extraction over addition:

$$dX = \mathrm{frac}\left(\mathrm{frac}(\alpha x/2) - \mathrm{frac}(y/2)\right), \tag{8}$$

which is equivalent to

$$dX = \mathrm{frac}\left(\mathrm{frac}(\tfrac{\alpha}{2}x) + \tfrac{1}{2}(y\%2)\right), \tag{9}$$

where % means modulo division. Thus, as far as dX is concerned, only $y\%2$ is relevant, rather than the actual value of y. We conclude that the first plot is simply a vertical tiling of the same pair of rows. Similarly, the second plot is a horizontal tiling of a pair of columns. This explains the role of the $\tfrac{1}{2}$ in Equation (1).

Let us zoom further into (9) to understand the structure of this pair of rows. First, notice that for the same value of x there is always a spacing of $\tfrac{1}{2}$ between the value of dX in even and odd rows. Now since t is always less than $\tfrac{1}{2}$, we conclude that for any x, if a point is set in the even row, then it is unset in the odd row, and if it is set in the odd row, then it
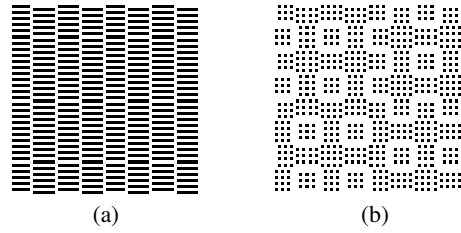

(a)   (b)

Figure 5: *(a) First-pass plot of (b) AA(58/31).*

is unset in the even row: it can not be set in both rows; though it can be unset in both rows, as we will see in a moment.

Next, notice that as we move along the *x*-axis, we increment by $\tfrac{\alpha}{2}$ to get consecutive values of dX. As long as only the fractional part is concerned, we are effectively *decrementing dX* by r with each step along the *x*-axis, as implied by (5).

Starting in the even row with some dX in the range

$$t - r \le dX < t,$$

we will have consecutive points set, with the "pen down" all the way till dX becomes less than r. This marks a dash in this row, and remember that the other (odd) row will remain unset during this interval.

The subsequent value of dX in the even row will cross the integer boundary and land somewhere within a distance of r below 1. That range is larger than t, so the pen will be up in the even row. In the odd row dX will be within a distance of r below $\tfrac{1}{2}$, which is still larger than t, so the pen will also be up. Thus, we have an unset point in both rows, which turns into an unset column in the first-pass plot.

Moving on along the *x*-axis, the subsequent value of dX will fall below t in the odd row, starting a dash in this row; and the process continues, with the even/odd roles swapped each time dX crosses the integer boundary.

The first-pass plot is illustrated in Figure 5. It is composed of a stack of dashes in the even-indexed rows, a one-pixel-thick gap, a stack of dashes in the odd-indexed rows, a one-pixel-thick gap, and so on. The plot appears like a vertical tiling of horizontally-running dashed lines, offset in such a way that dashes in one row are next to gaps in the adjacent rows. Dashes are one pixel shorter than gaps on each end, so that dashes in adjacent rows do not touch. These gap-pixels create vertically-running lines of 'voids' interspersed on the horizontally-running lines of dashes.

Another way to look at it is as columns of horizontal dashes at each second row. Adjacent columns are horizontally spaced by a one-pixel-thick void, and are vertically offset by one pixel, so that dashes of one column align with gaps in the adjacent two columns.

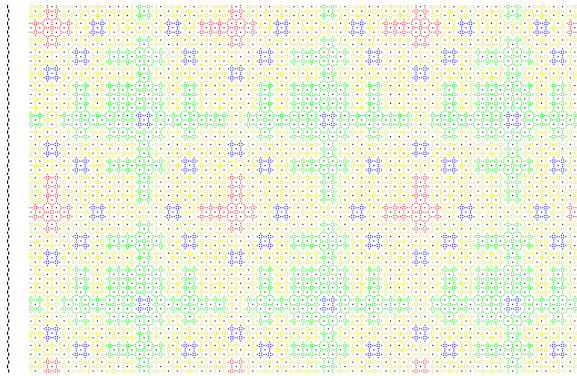The second-pass plot is essentially a 90 degrees rotation

Figure 6: *"Urea", the first synthesized AA Pattern. This pattern was not generated by Algorithm 1; it was generated by the superposition of two plots of vertically- and horizontally-running dashes of arbitrarily selected lengths and distributions.*

Figure 7: *Superposition of dashes.*

of the first-pass plot, composed of horizontally-running lines of voids interspersed on vertically-running lines of dashes. Or, seen from another perspective, it is composed of rows of vertical dashes in each-second column, with one-pixel-thick voids between rows.

4. Superposition

The two plots are combined with an 'and' operation, so that a pixel is set if and only if it is set in both plots. As illustrated in Figure 6, any such superposition of two plots like those described in Section 3 above will generate a pattern similar to AA Patterns, even if the plots were not coming from Algorithm 1.

When a column of horizontal dashes crosses a row of vertical dashes, the result is a grid of points separated horizontally and vertically by one pixel, as shown in Figure 7. They are these grids that constitute the general distribution of pixels in AA Patterns.

Notice that the lines of voids in one of the constituent plots run in the same direction as the lines of dashes in the other plot, so each void line will completely erase a dashed line. Erasing gaps in the dashed line makes no change, but erasing dashes has an interesting effect: dashes have adjacent gaps in both sides, so when a dash is erased the result is a three-pixel-thick gap in place of the erased dash. Thus, lines of voids will be replaced by intermittent lines of 'bold' gaps, as illustrated in Figure 8.
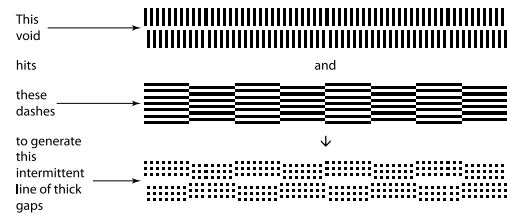

Figure 8: *How lines of voids in one plot interact with lines of dashes in the other plot. Lines of voids were ignored in the middle plot for sake of clarity.*

At each intersection of the void lines in the constituent plots, there is exactly one horizontal bold gap running either to the left or to the right of the intersection, depending on the location of the erased dash, which in turn depends on whether the index of the erased dashed line is even or odd. Similarly, at each intersection there is exactly one vertical bold gap running either up or down. Thus, whether the index of a horizontal or vertical void is even or odd, decides between two alternatives for the generated intermittent line of bold gaps.

How is that index defined, then? Well, the distance between consecutive void lines is the length of the dash between them, which is either $\lfloor t/r \rfloor$, or $\lfloor t/r \rfloor + 1$, depending on the value of dX at the beginning of the dash. Inevitably, one of these lengths is even and the other is odd. When the length of the dash is odd, the index of the following void line is the same as that of the preceding one in an even/odd sense; and the bold gaps will be aligned on these lines. When the dash length is even, the indexes of the two void lines are different: one is even and the other is odd; and the bold gaps will be offset.

5. AA Fractal

They are these intermittent lines of bold gaps that make the visible outlines of AA Patterns, inside which grids of the constituent points of the pattern are laid. Let us make a closeup of these outlines, abstracting them as dashed lines. There are only two possibilities for each dashed line, which strongly suggests a binary encoding, as we will use in our subsequent diagrams.

We will look at different configurations with the help of diagrams in Figure 9. Diagram (a) shows the simplest possible setup, in which all vertical and horizontal lines are aligned. This all-zeros configuration creates a matrix of small square 'islands', each outlined by a single dash on each side. (b) Shows the effect of toggling a single horizontal line: vertical dashes link dashes in the toggled line to dashes in an adjacent horizontal line, to form a first-order horizontal 'chain'; we will see higher orders soon. In (c) we toggle a single vertical line to make a first-order vertical chain. The two chains break at their point of intersection, and cross-link
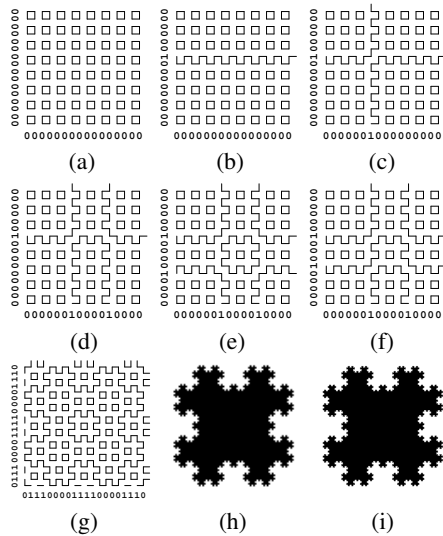
Figure 9: *Interaction of horizontal and vertical dashed lines.*

to make two bent chains. In this specific setup, the lower vertical chain is linked to the rightmost horizontal chain, and the upper to the leftmost.

In (d) we skip four vertical lines to the right, and toggle the fifth to create another vertical chain, which breaks on intersecting the horizontal chain, this time cross-linking lower-left and upper-right. At this point we have a 'peninsula' bounded from left, top, and right sides. (e) Toggling a second horizontal line, five steps below the previously toggled one, turns the peninsula into an island bounded by first-order chains. This island is larger that those in (a), and it houses a small island inside. (g) shows formation of a level-3 island in $AA(\sqrt{3})$ (the green cluster in Figure 1-(a)). (h) and (i), respectively, show level 4 and 5 islands from the same pattern, scaled to the same size, with inner islands removed to avoid distraction. Each island has a square 'mainland' and many peninsula which closely resemble the smaller-sized islands (along with their peninsulas).

If in (e) we were to toggle the fourth line instead of the fifth, then the vertical chains would have linked the new horizontal chain to the previous one to create a second-order chain, as shown in (f). Second-order chains, in turn, can be linked into higher order chains. Those chains are self-similar in the sense that they all resemble the first-order chain, but instead of dashes they have lower order chains as their links. Note that chains can vary in number of links.

To what extent can the described (highest-order) chains and (largest) islands be called fractals; comparing them to Koch curve and Koch island, for example? Well, in many aspects AA Patterns have an opposite sense to fractals; though they are, to some extent, equivalent. For example, elements

in Koch curve are scaled down while end points are fixed; whereas elements in AA chains are fixed while end points move apart. We can see the mainland of a Koch island because it has a finite area, but we can not see its coastline because it has infinitely fine detail; in contrast, we can not see the mainland of an infinitely large AA island, but we can see its coastline.

6. Conclusion

In this paper we investigated the fractal behaviour exhibited by AA Patterns. Our investigation revealed a form which we called "AA Fractal". It comes from the interaction of horizontal and vertical dashed lines, which cross in such a way that at each intersection there are exactly one horizontal and one vertical dashes.

The results we have found, and the way we arrived there, pose many questions, and open many doors for work to follow this paper. For example, what are the different properties and fractal measures of this AA Fractal? How do these measures, if any, relate to the pattern parameter? The parameter α encapsulates the recursive/iterative logic to generate AA Patterns; can the logic of different kinds of fractals be encapsulated similarly in real parameters? How can we explain the fractal-like shape of the coloring maps of AA Patterns?

At the application level, an important contribution of this paper is the two-plot decomposition of AA Patterns. This opens the door for synthesizing patterns with higher level of control over details. Another hint is the binary encoding of the outlines, which might suggest applications in visualization. Applications developed by the author will be available at http://aapatterns.abdallagafar.com

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