Wavejets: A Local Frequency Framework for Shape Details Amplification
Supplementary Material

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This supplementary material gives the mathematical proofs for the various theorems and corollaries.

1 Proof of the wavejets decomposition

Equation 1 of the paper contains terms such as $x^{k-j}y^j$, which can be rewritten as linear combinations of powers of $e^{i\theta}$.

\[ x^{k-j}y^j = r^k \cos^{k-j} \theta \sin^j \theta \]

\[ = r^k \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right)^{k-j} \left( \frac{e^{i\theta} - e^{-i\theta}}{2i} \right)^j \]

\[ = \frac{r^k}{2^{k+j}} \left( \sum_{l=0}^{k-j} \binom{k-j}{l} e^{(k-j-2l)i\theta} \right) \left( \sum_{l=0}^j \binom{j}{l} (-1)^l e^{(j-2l)i\theta} \right) \]

\[ = \frac{r^k}{2^{k+j}} \sum_{l_1=0}^{k-j} \sum_{l_2=0}^j (-1)^{l_1} \binom{k-j}{l_1} \binom{j}{l_2} e^{(k-2l_1-2l_2)i\theta} \]

\[ = \frac{r^k}{2^{k+j}} \sum_{l=0}^k \sum_{h=0}^j (-1)^{l} \binom{k-j}{h} \binom{j}{l-h} (-1)^h e^{i\theta} \]

\[ = r^k \sum_{n=-k}^k b(k,j,n) e^{in\theta} \quad (1) \]

with $b(k,j,n) = 0$ if $k$ and $n$ do not have the same parity and $b(k,j,n) = \frac{1}{2^{k+j}} \sum_{h=0}^{n-k} \binom{n-k}{h} \binom{j}{n-k-h} (-1)^h$ otherwise.

Using Equations 2 of the paper we get:

\[ \phi_{k,n} = \sum_{j=0}^k \frac{b(k,j,n)}{j!(k-j)!} f_{x^{k-j}y^j}(0,0). \quad (2) \]

2 Proof of the stability theorem (theorem 1)

Let us first recall the setting of this theorem. Let us call $T(p)$ the true tangent plane and $P(p)$ the chosen parameterization plane, also passing through $p$. One can find an axis $(p,u)$ and angle $\gamma$ such that the rotation of axis $(p,u)$ and angle $\gamma$ transforms $P(p)$ into $T(p)$. Since $p$ belongs to both planes, $(p,u)$ is aligned with line $T(p) \cap P(p)$. Let us parameterize $T(p)$ and $P(p)$ so that a point of the surface has
coordinates \((x = r \cos \theta, y = r \sin \theta, h)\) over \(T(p)\) and \((x = R \cos \Theta, y = R \sin \Theta, H)\) over \(P(p)\). Let us first assume that \(\theta\) (resp. \(\Theta\)) corresponds to the angular coordinate of point \(q\) with respect an origin vector aligned with \(u\) in \(T(p)\) (resp. with \(u\) in \(P(p)\)). We will state our main theorem in this setting and the generalization will follow naturally. In this setting the surface wavejets decomposition at point \(q\) writes

\[\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \phi_{k,n} r^k e^{i(n\mu - \phi)}\]

over the tangent plane \(T(p)\) and as \(\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \Phi_{k,n} r^k e^{i(n\mu - \phi)}\) over \(P(p)\). We can express the \(\Phi_{k,n}\) coefficients with respect to \(\phi_{k,n}\) and the rotation angle \(\gamma\). To generalize the theorem to arbitrary origin vectors for the angular coordinate in \(T(p)\) and \(P(p)\) for \(\theta\) and \(\Theta\), recall that a change of reference vector in \(T(p)\) amongs to a phase shift \(\mu\), one can always change the origin vector, compute the wavejets coefficients \(\phi_{k,n}\) and recover the real wavejets coefficients as \(\phi_{k,n} e^{i\mu\phi}\) (similar formulas hold for \(\Phi_{k,n}\)).

**Theorem 1.** The new coefficients \(\Phi_{k,n}\) can be expressed with respect to the \(\phi_{k,n}\) as follows:

\[
\begin{align*}
\Phi_{0,0} &= 0 \\
\Phi_{1,1} &= \Phi_{1,-1} = \frac{\gamma}{2} e^{-i\frac{\pi}{2}} + o(\gamma) \\
\Phi_{k,n} &= \phi_{k,n} + \gamma F(k,n) + o(\gamma)
\end{align*}
\]

**Proof.** The rotation matrix \(R\) of axis \(u = (1,0,0)\) and angle \(\gamma\) transforms the coordinates \((X,Y,H)\) of a surface point \(p\) in the parameterization of \(P(p)\) into coordinates \((x,y,h)\) in the parameterization of \(P(p)\). Let us assume that \(\gamma^2\) is small enough. Then the rotation has the following expression:

\[
R = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -\gamma \\
0 & u_\gamma & 1
\end{pmatrix} + o(\gamma)
\]

Thus, relation between \((x,y,f(x,y) = h)\) and \((X,Y,F(X,Y) = H)\) is the following:

\[
\begin{align*}
x &= X + o(\gamma) \\
y &= Y - \gamma H + o(\gamma) \\
h &= \gamma Y + H + o(\gamma)
\end{align*}
\]

Let us switch to polar coordinates \((r,\theta)\) (resp. \((R,\Theta)\)) such that \(x = r \cos \theta\) and \(y = r \sin \theta\) (resp. \(X = R \cos \Theta\) and \(Y = \sin \Theta\)). Let \(z = x + iy\) and \(Z = X + iY\). Equation (5) yields:

\[
h = H + \gamma RT(\Theta) + o(\gamma)
\]

With \(T(\Theta) = \frac{1}{2} \left(e^{i(\Theta - \frac{\pi}{2})} + e^{-i(\Theta - \frac{\pi}{2})}\right)\).

The following equation for \(r\) follows from \(z = x + iy\) and Equation 5:

\[
r^k = \sqrt{\frac{r}{z}}^k = R^k + \frac{kR^{k-1}H}{2} (e^{i(\Theta + \frac{\pi}{2})} + e^{-i(\Theta + \frac{\pi}{2})}) + o(\gamma)
\]

Similarly, we have for all \(n \in \mathbb{N}\):

\[
z^n = R^n e^{i\Theta} + nR^{n-1}H e^{i((n-1)\Theta + \frac{\pi}{2})} + o(\gamma)
\]

which yields, since \(e^{i\Theta} = (z/|z|)^n = (z/r)^n:\)

\[
e^{i\Theta} = e^{i\Theta} + \frac{nH}{2R} (e^{i((n-1)\Theta + \frac{\pi}{2})} - e^{i((n+1)\Theta + \frac{\pi}{2})}) + o(\gamma)
\]

Combining Equations 7 and 9, and setting \(A_{k,n} = \frac{(k+n)}{2} e^{-i\frac{\pi}{2}}\) yields:

\[
r^k e^{i\Theta} = R^k e^{i\Theta} + R^{k-1} e^{i\Theta} H \left(A_{k,n} e^{-i\Theta} + A_{k,-n} e^{i\Theta}\right) + o(\gamma)
\]

Plugging Equation 10 in Equation 6, one has:
A similar computation yields:

\[
H = \left( \sum_{k=0}^{\infty} \sum_{n=-k}^{n} \phi_{k,n} R^k e^{i\theta} \right) - \gamma RT(\Theta) + o(\gamma)
\]

Finally:

\[
F(\Theta) = \sum_{k=0}^{\infty} \sum_{n=-k}^{n} \phi_{k,n} R^k e^{i\theta} - \gamma RT(\Theta) + G(\Theta) + o(\gamma)
\]

With:

\[
F(\Theta) = \left( \sum_{k=0}^{\infty} \sum_{n=-k}^{n} \phi_{k,n} R^k e^{i\theta} \right)
\]

\[
G(\Theta) = \left( \sum_{k=0}^{\infty} \sum_{n=-k}^{n} \phi_{k,n} R^k e^{i\theta} \right)
\]

Recall that if \( k \) and \( n \) do not share the same parity, \( \phi_{k,n} = 0 \), then if \( m = -j - 1 \), \( \phi_{j+1,m+1} = 0 \). Furthermore by definition of \( A \), if \( m = -j + 2 \) then \( A_{j+1,m+1} = 0 \). Thus we can write:

\[
F(\Theta) = \left( \sum_{k=0}^{\infty} \sum_{n=-k}^{n} \phi_{k,n} R^k e^{i\theta} \right) \left( \sum_{j=0}^{\infty} \sum_{m=-j}^{m} \phi_{j+1,m} A_{j+1,m} R^j e^{i\theta} \right)
\]

Finally:

\[
F(\Theta) = \sum_{k=0}^{\infty} \sum_{n=-k}^{n} \phi_{k,n} R^k e^{i\theta} - \gamma RT(\Theta) + G(\Theta) + o(\gamma)
\]
For $k > 1$, one has the following relationship:

$$\Phi_{k,n} = \phi_{k,n} + \gamma \sum_{j=0}^{k-2} \sum_{p+m=n \atop |p| \leq k-j \atop |m| \leq j} \phi_{k-j,p}(\phi_{j+1,m+1}A_{j+1,m+1} + \phi_{j+1,m-1}A_{j+1,-m+1}^*) + o(\gamma)$$

(17)

$$= \phi_{k,n} + \gamma F(k,n) + o(\gamma)$$

\[ \square \]

3 Proof of Corollary 1

**Corollary 1.** It follows from Theorem 1 that $\gamma = 2|\Phi_{1,1}| + o(\gamma)$ and $\arg(\Phi_{1,1}) = \frac{\pi}{2} + o(\gamma)$. Thus if the rotation is small enough, it is possible to correct the parameterization by performing a rotation along axis $(1,0,0)$ with rotation angle $2|\Phi_{1,1}|$.

**Proof.** From Theorem 1, we have $\Phi_{1,1} = \frac{2}{\gamma}e^{-i\gamma} + o(\gamma)$. Then $|\Phi_{1,1}| = \gamma/2 + o(\gamma)$ and $\arg\Phi_{1,1} = -\frac{\pi}{2} + o(\gamma)$. To recover the tangent plane, one has thus to perform a rotation of angle $2|\Phi_{1,1}|$ around the rotation axis $(p,u)$.

\[ \square \]

4 Proof of Corollary 2

**Corollary 2.** One can recover the true coefficients $\phi_{k,n}$ iteratively by the following formula:

$$\phi_{k,n} = \Phi_{k,n} - \gamma \sum_{j=1}^{k-2} \sum_{p+m=n \atop |p| \leq k-j \atop |m| \leq j} \phi_{k-j,p}(\phi_{j+1,m+1}A_{j+1,m+1} + \phi_{j+1,m-1}A_{j+1,-m+1}^*) + o(\gamma)$$

(18)

In particular, $\phi_{2,0} = \Phi_{2,0} + o(\gamma)$, $\phi_{2,2} = \Phi_{2,2} + o(\gamma)$ and $\phi_{2,-2} = \Phi_{2,-2} + o(\gamma)$ which means that the mean curvature is also stable in $o(\gamma)$.

**Proof.** Let us rewrite Equation 17 as:

$$\phi_{k,n} = \Phi_{k,n} - \gamma \sum_{j=1}^{k} s_{j,k,n} + o(\gamma)$$

(19)

- For $j = 0$, $s_{0,k,n} = \phi_{0,k,n}(\phi_{1,1}A_{1,1} + \phi_{1,-1}A_{1,-1}^*)$ since $\phi_{1,1} = \phi_{1,-1} = 0$.
- For $j = k-1$, $s_{k-1,k,n} = \phi_{k-1,k,n}A_{k,n} + \phi_{k,n-2}A_{k,n-2}^* = 0$ since $\phi_{1,1} = 0$
- For $j = k$, $s_{k,k,n} = \phi_{0,0}(\phi_{k+1,n+1}A_{k+1,n+1} - k + 1, n + 1 + \phi_{k+1,n-1}A_{k+1,-n+1}^*) = 0$ since $\phi_{0,0} = 0$

Equation 17 thus yields:

$$\phi_{k,n} = \Phi_{k,n} - \gamma \sum_{j=1}^{k-2} \sum_{p+m=n \atop |p| \leq k-j \atop |m| \leq j} \phi_{k-j,p}(\phi_{j+1,m+1}A_{j+1,m+1} + \phi_{j+1,m-1}A_{j+1,-m+1}^*) + o(\gamma)$$

(20)

One can notice that all $\phi_{j,p}$ coefficients appearing in the sum are such that $l < k$. The correction procedure is straightforward: assuming we have corrected all $\Phi_{l,n}$ for all $l < k$ and $-l \leq n \leq l$ and have therefore access to $\phi_{l,n}$ for all $l < k$ and $-l \leq n \leq l$, up to some error in $o(\gamma)$, one can use Equation 20 to correct coefficients $\Phi_{k,n}$ for all $-k \leq n \leq k$.

\[ \square \]