Tutorial

Inverse Spectral Geometry

2/4

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Outline

- Laplacian eigenvalues and eigenvectors
- Fourier analysis on manifolds
- Functional maps
- Shape difference operators
- Shape-from-operator inverse problems
Entdeckungen
über die
Theorie des Klanges

von
Ernst Florens Friedrich Chladni,
der Philosophie und Rechte Docter zu Wittenberg.

Mit elf Kupferstichen.

Leipzig,
bei Weidmanns Erben und Reich.
1787.

Ernst Chladni (1756-1827)
Interpretation of Chladni plates

Behavior of waves on the plate is guided by the wave equation

\[ f_{tt}(x, t) = -\Delta f(x, t) \]
Behavior of waves on the plate is guided by the wave equation

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Separation of variables: solution of the form $f(x, t) = \varphi(x)\tau(t)$
Behavior of waves on the plate is guided by the wave equation

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**Separation of variables:** solution of the form \( f(x, t) = \varphi(x)\tau(t) \)

Plugging into the equation

\[ \varphi(x)\tau_{tt}(t) = -\Delta \varphi(x)\tau(t) \]
Interpretation of Chladni plates

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\[ \frac{\tau_{tt}(t)}{\tau(t)} = -\frac{\Delta \varphi(x)}{\varphi(x)} \]
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**Separation of variables:** solution of the form \( f(x, t) = \varphi(x) \tau(t) \)

Plugging into the equation

\[ \frac{\tau_{tt}(t)}{\tau(t)} = -\frac{\Delta \varphi(x)}{\varphi(x)} = -\lambda \]
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Plugging into the equation

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Spatial part of the solution satisfies the Helmholtz equation

\[ \Delta \varphi(x) = \lambda \varphi(x) \]
\[ \lambda = \text{vibration frequencies} \]
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\[ \phi = \text{vibration modes} \]
Riemannian geometry in one minute

- Manifold $\mathcal{X} =$ topological space
- No global Euclidean structure
- **Tangent plane** $T_x \mathcal{X} =$ local Euclidean representation of manifold $\mathcal{X}$ around $x$
Riemannian geometry in one minute

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- **No global Euclidean structure**
- **Tangent plane** $T_x \mathcal{X} = \text{local Euclidean representation of manifold } \mathcal{X} \text{ around } x$
- **Riemannian metric**

$$\langle \cdot, \cdot \rangle_x : T_x \mathcal{X} \times T_x \mathcal{X} \to \mathbb{R}$$

depending smoothly on $x$
Riemannian geometry in one minute

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- **Isometry** $= \text{metric-preserving shape deformation}$
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  \]
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- **Isometry** = metric-preserving shape deformation
- **Intrinsic** = expressible solely in terms of Riemannian metric
Calculus on manifolds: scalar and vector fields

- **Scalar field** \( f : \mathcal{X} \to \mathbb{R} \)
Calculus on manifolds: scalar and vector fields

- Scalar field \( f : \mathcal{X} \rightarrow \mathbb{R} \)
- Vector field \( F : \mathcal{X} \rightarrow T\mathcal{X} \)
Calculus on manifolds: scalar and vector fields

- **Scalar field** $f : \mathcal{X} \to \mathbb{R}$
- **Vector field** $F : \mathcal{X} \to T\mathcal{X}$
- **Hilbert space** with inner products

\[
\langle f, g \rangle_{\mathcal{F}(\mathcal{X})} = \int_{\mathcal{X}} f(x)g(x)dx
\]

\[
\langle F, G \rangle_{\mathcal{F}(T\mathcal{X})} = \int_{\mathcal{X}} \langle F(x), G(x) \rangle_x dx
\]

where $dx = \text{area element induced by the Riemannian metric}$
Calculus on manifolds: gradient

- **Differential** $df : T\mathcal{X} \to \mathbb{R}$ acting on vector fields
  
  $$df(x) = \langle \nabla f(x), \cdot \rangle_x$$
Calculus on manifolds: gradient

- **Differential** $df : T\mathcal{X} \to \mathbb{R}$ acting on vector fields
  
  $$df(x) = \langle \nabla f(x), \cdot \rangle_x$$

- **Directional derivative**
  
  $$df(x)F(x) = \langle \nabla f(x), F(x) \rangle_{T_x\mathcal{X}}$$

  “how much $f$ changes at $x$ in direction $F(x)$”
Calculus on manifolds: gradient

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df(x) = \langle \nabla f(x), \cdot \rangle_x
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"how much \( f \) changes at \( x \) in direction \( F(x) \)"

- **Intrinsic gradient operator**

\[
\nabla f : \mathcal{F}(\mathcal{X}) \to \mathcal{F}(T\mathcal{X})
\]

"direction of steepest change of \( f \)"
Intrinsic divergence operator

\[ \text{div} : \mathcal{F}(T\mathcal{X}) \rightarrow \mathcal{F}(\mathcal{X}) \]

“net flow of field \( F \) at \( x \)”
Calculus on manifolds: divergence

- **Intrinsic divergence** operator
  \[
  \text{div} : \mathcal{F}(T\mathcal{X}) \to \mathcal{F}(\mathcal{X})
  \]
  “net flow of field $F$ at $x$”

- Formal adjoint of the gradient
  \[
  \langle F, \nabla f \rangle_{\mathcal{F}(T\mathcal{X})} = \langle \nabla^* F, f \rangle_{\mathcal{F}(\mathcal{X})} = \langle -\text{div} F, f \rangle_{\mathcal{F}(\mathcal{X})}
  \]
Calculus on manifolds: Laplacian

- **Laplacian** $\Delta : \mathcal{F}(\mathcal{M}) \rightarrow \mathcal{F}(\mathcal{M})$
  
  \[ \Delta f = -\text{div}(\nabla f) \]

  “difference between $f(x)$ and average value of $f$ around $x$”
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- **Isometry-invariant**
- **Self-adjoint** $\langle \Delta f, g \rangle_{\mathcal{F}(\mathcal{X})} = \langle f, \Delta g \rangle_{\mathcal{F}(\mathcal{X})}$
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- **Positive semidefinite**
Calculus on manifolds: Laplacian

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- **Positive semidefinite** $\Rightarrow$ non-negative eigenvalues
Laplacian eigenfunctions: Euclidean

First eigenfunctions of 1D Euclidean Laplacian = standard Fourier basis
Laplacian eigenfunctions: non-Euclidean

First eigenfunctions of a manifold Laplacian = Fourier basis on manifolds
Laplacian eigenfunctions: non-Euclidean

For shapes with simple spectrum, Laplacian eigenfunctions are invariant (up to sign) to isometric deformations, $\psi_i = \pm T \phi_i$
Laplacian eigenbasis

- $\phi_1, \phi_2, \ldots$ is an **orthogonal basis** on $L^2(\mathcal{X})$, i.e. $\langle \phi_i, \phi_j \rangle_{L^2(\mathcal{X})} = \delta_{ij}$

- Smoothest orthogonal basis, due to minimization of the **Dirichlet energy**

\[
\min_{\phi_i} \| \nabla \phi_i \|^2 \quad \text{s.t.} \quad \| \phi_i \| = 1, \quad i = 1, 2, \ldots
\]

- $\phi_i \perp \text{span}\{\phi_1, \ldots, \phi_{i-1}\}$

- Optimal basis for smooth signals

- **Intrinsic**, hence invariant under inelastic deformations (isometries)

- Non-Euclidean analogy of the **Fourier transform**

Aflalo et al. 2013, 2015
Fourier analysis (Euclidean spaces)

A (square-integrable) function \( f : [-\pi, \pi] \to \mathbb{R} \) can be written as Fourier series

\[
f(x) = \sum_{\omega} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi) e^{i\omega \xi} d\xi \ e^{-i\omega x}
\]

\[= \hat{f}_1 + \hat{f}_2 + \hat{f}_3 + \ldots\]
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\[
\hat{f}_\omega = \langle f, e^{-i\omega x} \rangle_{L^2([-\pi, \pi])}
\]

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Fourier analysis (non-Euclidean spaces)

A (square-integrable) function \( f : \mathcal{X} \rightarrow \mathbb{R} \) can be written as Fourier series

\[
f(x) = \sum_{k \geq 1} \int_{\mathcal{X}} f(\xi) \phi_k(\xi) d\xi \quad \phi_k(x)
\]

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\hat{f}_k = \langle f, \phi_k \rangle_{L^2(\mathcal{X})}
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Heat diffusion on manifolds

**Heat equation** governs diffusion processes on manifolds:

\[
\begin{align*}
    f_t(x, t) &= -\Delta f(x, t) \\
    f(x, 0) &= f_0(x) \quad \text{initial condition}
\end{align*}
\]
Heat diffusion on manifolds

**Heat equation** governs diffusion processes on manifolds:

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Solution of the heat equation expressed through the heat operator

\[
f(x, t) = e^{-t\Delta} f_0(x)
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f(x, t) = e^{-t\Delta} f_0(x) = \sum_{k \geq 1} e^{-t\lambda_k} \langle f_0, \phi_k \rangle_{L^2(\mathcal{X})} \phi_k(x)
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under the **heat kernel** \( h_t(x, \xi) \)
Heat diffusion on manifolds

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\]

spectral filter \( \hat{p} \cdot \hat{f}_k \)

\[
= \int_X f_0(\xi) \sum_{k \geq 1} e^{-t\lambda_k} \phi_k(x) \phi_k(\xi) \, d\xi
\]

heat kernel \( h_t(x, \xi) \)
Heat kernel

Heat kernel $h_t(x, \cdot)$ at different points on a manifold
$$\lambda = \text{frequency}$$

$$\phi = \text{Fourier atoms}$$
\[ \lambda = \text{frequency} \]

\[ \phi = \text{Fourier atoms} \]
What can be recovered from the Laplacian spectrum?

**Heat trace** expansion

\[
\text{trace}(e^{-t\Delta}) = \sum_{k \geq 1} e^{-t\lambda_k} = \sum_{k \geq 1} a_k t^k
\]

for a 2-manifold (without boundary) allows to recover the following properties from the Laplacian spectrum:

Minakshisundaram, Pleijel 1949
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- **Area**: \( a_0 = \text{area}(\mathcal{X}) \)

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- **Total Gaussian curvature**: \( a_1 = \frac{1}{3} \int_{\mathcal{X}} K(x) dx \)

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for a 2-manifold (without boundary) allows to recover the following properties from the Laplacian spectrum:

- **Area:** \( a_0 = \text{area}(\mathcal{X}) \)

- **Total Gaussian curvature:** \( a_1 = \frac{1}{3} \int_{\mathcal{X}} K(x) dx \), and by Gauss-Bonnet theorem, the **Euler characteristic:** \( a_1 = \frac{2\pi}{3} \chi(\mathcal{X}) \)

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- ...
Laplacian spectrum used as deformation-invariant shape descriptor

Reuter, Wolter, Peinecke 2005
isometric $\iff$ isospectral
isometric ⇐ ? ⇒ isospectral
CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

"La Physique ne nous donne pas seulement l’occasion de résoudre des problèmes . . . , elle nous fait présenter la solution." H. POINCARÉ.
Counter-examples

One cannot hear the shape of the drum!

Gordon, Web, Wolpert 1992
\[ \lambda = \text{frequency} \]

\[ \phi = \text{Fourier atoms} \]
Pointwise correspondence

Point-wise map $\tau: \mathcal{X} \rightarrow \mathcal{Y}$
Functional correspondence

Functional map $T: \mathcal{F}(\mathcal{X}) \rightarrow \mathcal{F}(\mathcal{Y})$

Ovsjanikov et al. 2012
Functional correspondence in spectral domain

\[ T \Rightarrow \hat{g} = C \hat{f} \]

Ovsjanikov et al. 2012
Functional correspondence in spectral domain

\[ f \approx \hat{f}_1 + \hat{f}_2 + \cdots + \hat{f}_k \]

\[ g \approx \hat{g}_1 + \hat{g}_2 + \cdots + \hat{g}_k \]

\[ \phi_1, \phi_2, \ldots, \phi_k \]

\[ \psi_1, \psi_2, \ldots, \psi_k \]

\[ \downarrow T \]

\[ \downarrow C \]

Ovsjanikov et al. 2012
Functional correspondence in spectral domain

Functional correspondence boils down to a linear equation w.r.t. $C$

$$g = Tf \iff \hat{g} = C\hat{f}$$

Ovsjanikov et al. 2012
Functional correspondence in spectral domain

\[ \hat{G}_{k \times q} = \langle \psi_i, g_j \rangle \mathcal{F}(\mathcal{Y}) \]

\[ \hat{C}_{k \times k} \]

\[ \hat{F}_{k \times q} = \langle \phi_i, f_j \rangle \mathcal{F}(\mathcal{X}) \]

where \( \hat{F}, \hat{G} \) are Fourier coefficients of corresponding ‘probe’ functions

\[ g_i \approx T f_i \quad i = 1, \ldots, q \geq k \]

Ovsjanikov et al. 2012
Functional maps: Summary

- Intrinsic by construction

- Operate with Fourier coefficients, forget about specific discretization (can apply to meshes, point clouds, etc.)

- Advantage: replace intractable combinatorial problems with tractable linear algebra problems

- Disadvantage: hard to guarantee properties of maps (e.g. bijectivity)

- Some properties can be guaranteed: e.g. area preservation = orthogonality of the functional map (\(C^T C = I\))
Shape difference operators

Distortions induced by a map = change of Riemannian metric
(inner product of tangent vectors)

Rustamov et al., 2013
Shape difference operators

$T \langle \cdot , \cdot \rangle_{\mathcal{F}(\mathcal{X})}$

$Tf \langle \cdot , \cdot \rangle_{\mathcal{F}(\mathcal{Y})}$

$Tg$

$\mathcal{F}(\mathcal{X})$

$\mathcal{F}(\mathcal{Y})$

Distortions induced by a map = change of inner products of functions

Rustamov et al., 2013
Shape difference operators

\[
\langle f, g \rangle_{\mathcal{F}(X)} \neq \langle Tf, Tg \rangle_{\mathcal{F}(Y)}
\]

Rustamov et al., 2013
Shape difference operators

\[ \langle f, g \rangle_{\mathcal{F}(X)} \neq \langle Tf, Tg \rangle_{\mathcal{F}(Y)} \]

**Riesz theorem**: there exists a unique self-adjoint linear operator

\[ D : \mathcal{F}(X) \rightarrow \mathcal{F}(X) \]

such that

\[ \langle Tf, Tg \rangle_{\mathcal{F}(Y)} = \langle f, Dg \rangle_{\mathcal{F}(X)} \]

Rustamov et al., 2013
Shape difference operators

- Captures the **difference** in the geometry of the two shapes
- **Depends** on choice of inner product

Rustamov et al., 2013
Discretization
Manifold meshes

Surface discretized as a embedded triangular mesh \((X, \mathcal{E}, \mathcal{F})\)

- **Nodes**: points in 3D \((n \times 3\) matrix \(X\) of coordinates)
- **Edges**: \((i, j)\), weighted by metric \(\ell_{ij} = \|x_i - x_j\|\)
- **Faces**: \((i, j, k)\) s.t. \((i, j), (i, k), (k, j) \in \mathcal{E}\)

Manifoldness assumption:
- Each edge is shared by two triangles
- Boundary of triangles incident on each node forms a single loop of edges
Laplacian discretization

Cotangent Laplacian $\Delta = A^{-1} W$ expressed in terms of discrete metric $\ell_{ij} = \|x_i - x_j\|$ where

$$w_{ij} = \begin{cases} \frac{-\ell_{ij}^2 + \ell_{jk}^2 + \ell_{ki}^2}{8A_{ijk}} + \frac{-\ell_{ij}^2 + \ell_{jh}^2 + \ell_{hi}^2}{8A_{ijh}} & \text{if } e_{ij} \in \mathcal{E}_i \\ \frac{-\ell_{ij}^2 + \ell_{jh}^2 + \ell_{hi}^2}{8A_{ijh}} & \text{if } e_{ij} \in \mathcal{E}_b \\ -\sum_{k \neq i} w_{ik} & \text{if } i = j \end{cases}$$

where $A_{ijk}$ is area of triangle $ijk$ and $a_i = \frac{1}{3} \sum_{ijk:i,j,k \in \mathcal{E}} A_{ijk}$

Meyer et al. 2003
Shape difference discretization

- **area-based**

\[
\langle f, g \rangle_{L^2(X)} = \int_X f(x)g(x)dx
\]

\[
D = V_{X,Y} = A_X^{-1}T^\top A_Y T
\]

Rustamov et al., 2013
Shape difference discretization

- **area-based**

\[
\langle f, g \rangle_{L^2(X)} = \int_X f(x)g(x)dx \\
D = V_{X,Y} = A_X^{-1}T^TA_YT
\]

- **conformal-based**

\[
\langle f, g \rangle_{H^1(X)} = \int_X \langle \nabla f(x), \nabla g(x) \rangle_x dx \\
D = R_{X,Y} = W_X^\dagger T^TW_YT
\]

Rustamov et al., 2013
Shape difference discretization

- **area-based**

\[ \langle f, g \rangle_{L^2(X)} = \int_X f(x)g(x)\,dx \]

\[ D = V_{X,Y} = A_X^{-1}T^\top A_Y T \]

- **V = I**: area-preserving maps

- **conformal-based**

\[ \langle f, g \rangle_{H^1(X)} = \int_X \langle \nabla f(x), \nabla g(x) \rangle_x \,dx \]

\[ D = R_{X,Y} = W_X^\dagger T^\top W_Y T \]

- **R = I**: angle-preserving (conformal) map

- **V = R = I**: isometric map

Rustamov et al., 2013
Shape-from-Operator: Recovering Shapes from Intrinsic Operators

Davide Boscaini, Davide Eynard, Drosos Kourounis, and Michael M. Bronstein
Università della Svizzera Italiana (USI), Lugano, Switzerland
Shape-from-Operator problems

Generic **Shape-from-Operator (SfO)** problem: given some intrinsic operator $O_0$, find an embedding $X$ by minimizing some cost function

$$
\min_{X \in \mathbb{R}^{n \times 3}} E(O(\ell(X)), O_0)
$$

Boscaini et al. 2014
Generic **Shape-from-Operator (SfO)** problem: given some intrinsic operator $O_0$, find an embedding $X$ by minimizing some cost function

$$
\min_{X \in \mathbb{R}^{n \times 3}} E(O(\ell(X)), O_0)
$$

$O$ depends on $X$ indirectly through the discrete metric $\ell(X)$, very hard for optimization!

Boscaini et al. 2014
Shape-from-Operator problems

- **Metric-from-Operator (MfO):** \( \min_{\ell} E(O(\ell), O_0) \) s.t. \( \Delta \) inequality

- **Shape-from-Metric (SfM):** \( \min_{X \in \mathbb{R}^{n \times 3}} \sum_{(i,j) \in \mathcal{E}} (\|x_i - x_j\| - \ell_{ij})^2 \),

Boscaini et al. 2014
Shape-from-Operator problems

- **Metric-from-Operator (MfO):** \( \min_{\ell} E(O(\ell), O_0) \) s.t. \( \Delta \) inequality

- **Shape-from-Metric (SfM):** 
  \[
  \min_{X \in \mathbb{R}^{n \times 3}} \sum_{(i,j) \in E} \left( \|x_i - x_j\| - \ell_{ij} \right)^2,
  \]

Boscaini et B 2014
Special setting of **MDS**: given a metric \( \ell \), find its Euclidean realization by minimizing the stress

\[
\min_{X \in \mathbb{R}^{n \times 3}} \sum_{i,j=1}^{n} v_{ij} \left( \| x_i - x_j \| - \ell_{ij} \right)^2,
\]

where

\[
v_{ij} = \begin{cases} 
1 & \text{if } (i, j) \in \mathcal{E}, \\
0 & \text{otherwise}
\end{cases}
\]

Leeuw et al., 1977
Shape-from-metric

Special setting of MDS: given a metric $\ell$, find its Euclidean realization by minimizing the stress

$$\min_{X \in \mathbb{R}^{n \times 3}} \sum_{i,j=1}^{n} v_{ij} (\|x_i - x_j\| - \ell_{ij})^2,$$

where

$$v_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{otherwise} \end{cases}$$

SMACOF algorithm: fixed point iteration of the form

$$X \leftarrow Z^\dagger B(X)X$$

where

$$Z = \begin{cases} -v_{ij} & \text{if } i \neq j, \\ \sum_{i \neq j} v_{ij} & \text{if } i = j \end{cases}$$

$$B(X) = \begin{cases} -\frac{v_{ij} \ell_{ij}}{\|x_i - x_j\|} & \text{if } i \neq j \text{ and } x_i \neq x_j, \\ 0 & \text{if } i \neq j \text{ and } x_i = x_j, \\ \sum_{i \neq j} b_{ij} & \text{if } i = j \end{cases}$$

Leeuw et al., 1977
From operators to shapes

- **Metric-from-Operator (MfO):** \( \min_{\ell} E(O(\ell), O_0) \) s.t. \( \Delta \) inequality

- **Shape-from-Metric (SfM):**
  \[
  \min_{X \in \mathbb{R}^{n \times 3}} \sum_{(i,j) \in \mathcal{E}} \left( \|x_i - x_j\| - \ell_{ij} \right)^2,
  \]

Boscaini et B 2014
Given a reference Laplacian operator $\Delta_Y$
Given a reference Laplacian operator $\Delta_γ$, and a corresponding initial shape $\mathcal{X}$, deform $\mathcal{X}$ by minimizing

$$\min_{X \in \mathbb{R}^{n \times 3}} \| A^{-1}(\ell(X)) W(\ell(X)) - \Delta_γ \|$$

Boscaini et al. 2014
Given a reference Laplacian operator $\Delta_Y$, and a corresponding initial shape $X$, deform $X$ by minimizing

$$\min_{X \in \mathbb{R}^{n \times 3}} \|A^{-1}(\ell(X))W(\ell(X)) - \Delta_Y\|$$

Boscaini et al. 2014
Given a reference Laplacian operator $\Delta_Y$, and an initial shape $X$ related by functional correspondence $T$, deform $X$ by minimizing

$$\min_{X \in \mathbb{R}^{n \times 3}} \| TA^{-1}(\ell(X)) W(\ell(X)) - \Delta_Y T \|$$

Boscaini et al. 2014
Shape-from-Laplacian convergence

Convergence of our method in the shape-from-Laplacian optimization problem using different initializations.

Boscaini et al. 2014
Style transfer by shape-from-Laplacian

“Modify $x$ such that $\Delta x$ becomes as similar as possible to reference Laplacian $\Delta y$”

Boscaini et al. 2014
“Modify $\mathcal{X}$ such that $\Delta_{\mathcal{X}}$ becomes as similar as possible to reference Laplacian $\Delta_{\mathcal{Y}}$”

Boscaini et al. 2014
Style transfer by shape-from-Laplacian

“Modify $\mathcal{X}$ such that $\Delta_{\mathcal{X}}$ becomes as similar as possible to reference Laplacian $\Delta_{\mathcal{Y}}$”
Style transfer by shape-from-Laplacian

“Modify $\mathcal{X}$ such that $\Delta_{\mathcal{X}}$ becomes as similar as possible to reference Laplacian $\Delta_{\mathcal{Y}}$”

Boscaini et al. 2014
Sensitivity to map quality

Functional map approximated as a matrix $T \approx \Psi_k C^\top \Phi_k^\top$ of rank $k$ using the first functions in Fourier expansion (larger $k$ = better map)

Ovsjanikov et al. 2012
Sensitivity to map quality

Functional map approximated as a matrix $\mathbf{T} \approx \Psi_k \mathbf{C}^T \Phi_k^T$ of rank $k$ using the first functions in Fourier expansion (larger $k$ = better map)

Shape-from-Laplacian result for different quality of the map $\mathbf{T}$
(initial shape: sphere, reference Laplacian: bumped sphere)

Ovsjanikov et al. 2012; Boscaini et B 2014
Shape-from-difference operator

Deform initial shape $\mathcal{X}$ to make it different from $\mathcal{C}$ same way as $\mathcal{B}$ is different from $\mathcal{A}$

$$
\min_{\mathbf{X} \in \mathbb{R}^{n \times 3}} \| D_{\mathcal{C},\mathcal{X}}(\ell(\mathbf{X})) T_{\mathcal{A},\mathcal{C}} - T_{\mathcal{A},\mathcal{C}} D_{\mathcal{A},\mathcal{B}} \|
$$

Boscoihi et al. 2014
Deform initial shape $\mathcal{X}$ to make it different from $\mathcal{C}$ same way as $\mathcal{B}$ is different from $\mathcal{A}$

$$\min_{X \in \mathbb{R}^{n \times 3}} \| D_{C,X}(\ell(X)) T_{A,C} - T_{A,C} D_{A,B} \|$$

Boscaini et B 2014
Shape-from-difference operator

Deform initial shape \( \mathcal{X} \) to make it different from \( \mathcal{C} \) same way as \( \mathcal{B} \) is different from \( \mathcal{A} \)

\[
\min_{\mathbf{x} \in \mathbb{R}^{n \times 3}} \mu \| \mathbf{A}_{C}^{-1} \mathbf{T}_{C, \mathcal{X}} \mathbf{A}(\ell(\mathbf{X})) \mathbf{T}_{C, \mathcal{X}} \mathbf{T}_{A, \mathcal{C}} - \mathbf{T}_{A, \mathcal{C}} \mathbf{V}_{\mathcal{A}, \mathcal{B}} \| + \\
(1 - \mu) \| \mathbf{W}_{\mathcal{C}}^{\dagger} \mathbf{T}_{C, \mathcal{X}} \mathbf{W}(\ell(\mathbf{X})) \mathbf{T}_{C, \mathcal{X}} \mathbf{T}_{A, \mathcal{C}} - \mathbf{T}_{A, \mathcal{C}} \mathbf{R}_{\mathcal{A}, \mathcal{B}} \|
\]

Boscaini et B 2014
Shape-from-difference operator

Deform initial shape \( \mathcal{X} \) to make it different from \( \mathcal{C} \) same way as \( \mathcal{B} \) is different from \( \mathcal{A} \)

\[
\min_{\mathcal{X}} \quad \mu \| A^{-1}_C A(\ell(\mathcal{X})) T_{A,C} - T_{A,C} V_{A,B} \| + \\
(1 - \mu) \| W^+_C W(\ell(\mathcal{X})) T_{A,C} - T_{A,C} R_{A,B} \|
\]

(initializing \( \mathcal{X} = \mathcal{C} \) we have \( T_{C,\mathcal{X}} = I \))

Boscaini et B 2014
Shape-from-difference convergence

Convergence of our method in the shape-from-difference optimization problem. Colors show vertex-wise MfO energy contribution.

Boscaini et al. 2014
Convergence of our method in the shape-from-difference optimization problem.

Boscaini et al. 2014
Analogy synthesis by shape-from-difference

“Find $\mathcal{X}$ such that the difference operator between $C, \mathcal{X}$ is as similar as possible to the given difference operator between $A, B$”

Boscaini et al. 2014
Analogy synthesis by shape-from-difference

“Find $\mathcal{X}$ such that the difference operator between $C$, $\mathcal{X}$ is as similar as possible to the given difference operator between $A$, $B$”

Boscaini et al 2014
Analogy synthesis by shape-from-difference

\[ \mathcal{X} = C + (B - A) \]

“Find \( \mathcal{X} \) such that the difference operator between \( C, \mathcal{X} \) is as similar as possible to the given difference operator between \( A, B \)”

Boscaini et B 2014
Shape exaggeration

\[ \mathcal{A} \quad \mathcal{B} \quad \mathcal{B} + (\mathcal{B} - \mathcal{A})' \quad \mathcal{B} + 2(\mathcal{B} - \mathcal{A})' \]

Shape exaggeration obtained by applying the difference operator between \(\mathcal{A}, \mathcal{B}\) to \(\mathcal{B}\) several times

Boscaini et B 2014
Summary

- Laplacian eigenvectors form an orthogonal basis for functions on a manifold
- Lifting to space of functions on manifolds: replace combinatorial problems with linear algebra problems
- Operate with Fourier coefficients, forget about specific discretization
- Represent shapes as operators (contains full information up to some class of transformations, e.g. isometries or conformal maps)
- Recover shapes from operators
- Interesting links to Geometric Deep Learning