Symmetry in Shapes – Theory and Practice

Intrinsic Symmetry Detection

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Intrinsic Symmetries
Intuition

I am symmetric.

What about us?

Bronstein et al.
Problem Formulation

- Shape $X$ is **symmetric**, if there exists a transformation $f$ such that $f(X) = X$.

What class of transformations is allowed?

- **Extrinsic:**
  - $f$ is a combination of:
    - Rotation,
    - Translation,
    - Reflection,
    - (Scaling)

Bronstein et al.
• Shape $X$ is \textbf{symmetric}, if there exists a transformation $f$ such that $f(X) = X$.

What class of transformations is allowed?

• \textbf{Extrinsic}:
  
  $f$ is: rotation, translation, reflection

• \textbf{Intrinsic}?

Bronstein et al.
Fundamental Theorem:

A map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a combination of translation, rotation, and reflection if and only if it preserves all Euclidean distances.
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\[ d_{\mathbb{R}^3}(f(x), f(y)) = d_{\mathbb{R}^3}(x, y) \quad \forall \, x, y \]
Extrinsic Formulation

- Shape $X$ is **extrinsically symmetric**, if there exists a rigid motion $f$, such that $f(X) = X$.

Equivalently:
- Shape $X$ is **extrinsically symmetric**, if there exists a map:
  
  $$f : X \rightarrow X \text{ s.t. } d_{\mathbb{R}^3}(f(x), f(y)) = d_{\mathbb{R}^3}(x, y) \quad \forall \ x, y$$

Bronstein et al.
Extrinsic Formulation

- Shape $X$ is **extrinsically symmetric**, if there exists a **rigid motion** $f$,
such that $f(X) = X$.

Equivalently:
- Shape $X$ is **extrinsically symmetric**, if there exists a map:
  
  \[
  f : X \rightarrow X \quad \text{s.t.} \quad d_{\mathbb{R}^3}(f(x), f(y)) = d_{\mathbb{R}^3}(x, y) \quad \forall \ x, y
  \]
Intrinsic Formulation

- Shape $X$ is **intrinsically symmetric**, if there exists a map:

  $$f : X \rightarrow X \text{ s.t.}$$

  $$d_X(f(x), f(y)) = d_X(x, y) \quad \forall \ x, y$$

Bronstein et al.
**Intrinsic Formulation**

- **Intrinsic Isometries:**
  Shape deformations that preserve intrinsic (geodesic) distances.
Intrinsic Formulation

- **Intrinsic Symmetries:**
  Self-maps that approximately preserve geodesic distances
Intrinsic Formulation

Extrinsic symmetries depend on the embedding of the object in space.

Intrinsic symmetries are defined with respect to an intrinsic metric of the surface.
Intrinsic Symmetry Detection

Idea:

1. Solve the optimization problem directly:

\[
\min_{f:X \to X} \sum_{x,x' \in X} (d_X(x, x') - d_X(f(x), f(x')))^2
\]

Approach:

**GMDS:** treat each point as a variable, solve using nonlinear optimization (main difficulty: obtaining the gradient of the energy).

Raviv et al., *Symmetries of Non-Rigid Shapes.*, NRTL 2007, IJCV 2009
Idea:

1. Solve the optimization problem directly:

\[
\min_{f: X \to X} \sum_{x, x' \in X} (d_X(x, x') - d_X(f(x), f(x')))^2
\]

Difficulties:

1. Energy is non-linear non-convex, need a good initial guess.
2. Optimization is expensive (compute over a small number of points).
3. Want to stay away from the trivial solution.

Raviv et al., *Symmetries of Non-Rigid Shapes.*, NRTL 2007, IJCV 2009
Intrinsic Symmetry Detection

Initial Guess:

1. Adapt the Global Rigid Matching idea to non-rigid setting:
   1. For each point on the surface find a **non-rigid descriptor**.
   2. Match points with similar descriptors.
   4. Compute the distortion of the partial solution.

2. Branch and bound global optimum
   1. Incrementally add points to get a partial solution.
   2. If the distortion is greater than the known solution, disregard it.
   3. Depends on the quality of the initial greedy guess.

Raviv et al., *Symmetries of Non-Rigid Shapes*, NRTL 2007, IJCV 2009
Non-rigid Descriptor:

1. At each point compute the histogram of geodesic distances.

Comparing Descriptors:

1. Non-trivial. Comparing $\|h_i - h_j\|_2$ bad because of binning. Use instead:

$$d(h_i, h_j) = \sqrt{(h_i - h_j)^T A(h_i - h_j)}$$

where $A_{mn}$ distance between bins.
Intrinsic Symmetry Detection

Results:

Limitations:

1. Not easy to explore *multiple* symmetries.
2. Need a better descriptor.
Intrinsic Symmetry Detection

Purely algebraic method for detecting intrinsic symmetries, and point-to-point correspondences.

Grouping symmetries into discrete classes.

Main Observation: In a certain space, intrinsic symmetries become extrinsic symmetries.

O., Sun, Guibas, *Global Intrinsic Symmetries of Shapes*, SGP 2008
Global Point Signatures

Given a point $x$ on the surface, its GPS signature:

$$s(x) = \left( \frac{\phi_1(x)}{\sqrt{\lambda_1}}, \frac{\phi_2(x)}{\sqrt{\lambda_2}}, \ldots, \frac{\phi_i(x)}{\sqrt{\lambda_i}}, \ldots \right)$$

Rustamov, 2007

Where $\phi_i(x)$ is the value of the eigenfunction of the Laplace-Beltrami operator at $x$. 
Given a compact Riemannian manifold $X$ without boundary, the Laplace-Beltrami operator:

$$\Delta : C^\infty(X) \to C^\infty(X), \Delta f = \text{div} \nabla f$$
Laplace-Beltrami Operator

Given a compact Riemannian manifold $X$ without boundary, the Laplace-Beltrami operator $\Delta :$

1. Is invariant under isometric deformations.

2. Characterizes the manifold completely.

3. Has a countable eigendecomposition:

$$\Delta \phi_i = \lambda_i \phi_i$$

that forms an orthonormal basis for $L^2(X)$. 
Observations

GPS($X$)

\[ s(x) = \left( \frac{\phi_1(x)}{\sqrt{\lambda_1}}, \frac{\phi_2(x)}{\sqrt{\lambda_2}}, \ldots, \frac{\phi_i(x)}{\sqrt{\lambda_i}}, \ldots \right) \]

Theorem:

If $X$ has an intrinsic symmetry $f : X \rightarrow X$, then GPS($X$) has a Euclidean symmetry. I.e.:

\[ \| s(x) - s(x') \|_2 = \| s(f(x)) - s(f(x')) \|_2 \ \forall \ x, x' \in X \]

Moreover, restriction to each distinct eigenvalue is symmetric.

O., Sun, Guibas, *Global Intrinsic Symmetries of Shapes*, SGP 2008
Observations

Theorem:

If \( M \) has an intrinsic symmetry \( T : M \to M \), then GPS(\( M \)) has a Euclidean symmetry. I.e.:

\[ \|s(x) - s(y)\|_2 = \|s(T(x)) - s(T(y))\|_2 \quad \forall x, y \in M \]

Moreover, restriction to each distinct eigenvalue is symmetric. For non-repeating eigenvalues, only 2 possibilities:

- Positive
  \[ \phi_i(T(x)) = \phi_i(x) \quad \forall x \in M \]

- Negative
  \[ \phi_i(T(x)) = -\phi_i(x) \quad \forall x \in M \]
Restricted Signature Space

- Only include non-repeating eigenvalues.

In the restricted space, intrinsic symmetries are reflective symmetries around principal axes:

\[ |s_i(f(x))| = |s_i(x)| \]

Detecting such symmetries is easy. Find which coordinates to flip. For each point \( x \), is there a correspondence \( y \), s.t.

\[ |s_j(y)| = |s_j(x)| \quad \forall j \neq i, \text{ and } s_i(y) = -s_i(x) \]

- Only need nearest neighbor computation in high \( d \).

ANN library for nearest neighbor computations.

Query in KD-tree depends on the dimension of the data. GPS is homeomorphism, so dimension = 2. No curse of dimensionality with increasing d.

Overall complexity $O(d^3 n \log n)$. 
Results

Euclidean symmetries when present.

Two different symmetries for human shape.
Topological Noise

Change in GPS after geodesic shortcuts:

Correspondences

Original  finger handle  Body-arm handle
Limitations

- Can only detect very global symmetries.
- Cannot handle continuous symmetries.
- In the discrete setting even non-repeating eigenfunctions can be unstable.

O., Sun, Guibas, *Global Intrinsic Symmetries of Shapes*, SGP 2008
Möbius Voting:

Isometries are a subgroup of the group of conformal maps. For genus zero surfaces: 3 correspondences constrain all degrees of freedom, and the optimal transformation has a closed form solution.

Lipman and Funkhouser SIGGRAPH’09
Möbius Voting for shape matching:

Isometries are a subgroup of the group of conformal maps. For genus zero surfaces: 3 correspondences constrain all degrees of freedom, and the optimal transformation has a closed form solution.
Möbius Voting-based symmetry detection:

1) Map the mesh surface to the extended complex plane $\hat{\mathbb{C}}$.

2) Generate a set of anti-Möbius transformations.

3) Measure alignment score.

4) Return the best alignment.

Iterate

Möbius Voting-based symmetry detection:

1) Map the mesh surface to the extended complex plane \( \mathbb{C} \).

Mid-point uniformisation (Lipman et al. ‘09)

Conformal mapping onto the sphere by solving a sparse linear (Laplacian) system

Kim, Lipman, Chen, and Funkhouser *Mobius Transformations for Global Intrinsic Symmetry Analysis*, SGP 2010
Möbius Voting-based symmetry detection:

1) Map the mesh surface to the extended complex plane $\mathbb{C}$.
2) Generate a set of anti-Möbius transformations.

Find likely triplets of correspondences

Use intrinsic symmetry-invariant descriptors.

Intrinsic Symmetry Detection

Möbius Voting-based symmetry detection:

1) Map the mesh surface to the extended complex plane \( \hat{\mathbb{C}} \).
2) Generate a set of anti-Möbius transformations.
3) Measure alignment score.

Use the initial triplet to find correspondences between all other points.

Möbius Voting-based symmetry detection:

1) Map the mesh surface to the extended complex plane $\hat{\mathbb{C}}$.
2) Generate a set of anti-Möbius transformations.
3) Measure alignment score. Use the initial triplet to find correspondences between all other points. Closed form solution in the extended complex plane embedding.

Kim, Lipman, Chen, and Funkhouser: Möbius Transformations for Global Intrinsic Symmetry Analysis, SGP 2010
Möbius Voting-based symmetry detection:

1) Map the mesh surface to the extended complex plane \( \hat{\mathbb{C}} \).
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Kim, Lipman, Chen, and Funkhouser *Mobius Transformations for Global Intrinsic Symmetry Analysis*, SGP 2010
Results

Largest-scale evaluation of an intrinsic symmetry-detection method.

Benchmark for comparing other methods.
Continuous Intrinsic Symmetries

Ben-Chen, Butscher, Solomon, Guibas On discrete killing vector fields and patterns on surfaces, SGP 2010
Continuous Intrinsic Symmetries

Ben-Chen, Butscher, Solomon, Guibas *On discrete killing vector fields and patterns on surfaces*, SGP 2010
Represent Transformations using Tangent Vector Fields

$\phi_t(p)$ – One-parameter family of mappings generated by the tangent vector field $U$
$g_p(X,Y)$
Preserve the Metric

\[ \lim_{t \to 0} \frac{g_{\phi^t(p)}(X', Y') - g_p(X, Y)}{t} = 0 \]
Killing Vector Fields

Vector fields whose flow preserves the metric $U$ is a KVF if for any $X, Y$:

$$\mathcal{L}_U g = \lim_{t \to 0} \frac{g_p(X, Y) - g_{\phi^t(p)}(X', Y')}{t} = 0$$
Killing Vector Fields (again)
The Killing Equation

- $\mathbf{U}$ is a KVF if and only if for every $\mathbf{V}$:

$$g_p \left( \nabla^\mathbf{V} \mathbf{U}, \mathbf{V} \right) = 0$$

- In $\mathbb{R}^n$ means:

$$\left\langle \nabla \mathbf{U} \cdot \mathbf{V}, \mathbf{V} \right\rangle = 0$$
Killing Vector Fields
A (very) simple example

\[ U = (u(x,y), v(x,y)) = (-y, x) \]
Back to the Killing Equation

\[ \forall V \quad \left\langle \nabla U \cdot V, V \right\rangle = 0 \]

Equivalent to:

\[ \nabla U + \nabla U^T = 0 \]

\( R^n : \) \quad \nabla U = \text{Jacobian matrix}

Surface: \quad \nabla U = \text{covariant derivative tensor}
Computing AKVFss

Solve:

\[
\min_U E_K(U) = \int_M \left| \nabla U + \nabla U^T \right|^2 \, dv \quad s.t. \quad \int_M |U|^2 \, dv = 1
\]

On a triangulated mesh.

Reformulate using (Discrete) Exterior Calculus. Leads to an eigendecomposition problem.
AKVFs in the Wild

Ben-Chen, Butscher, Solomon, Guibas *On discrete killing vector fields and patterns on surfaces*, SGP 2010
Approximate KVF

Noise

\[ \sigma = 0.065 \quad E = 0.29 \]
\[ \sigma = 0.087 \quad E = 0.55 \]
\[ \sigma = 0.1145 \quad E = 1.33 \]
\[ \sigma = 0.2 \quad E = 6.7 \]

Ben-Chen, Butscher, Solomon, Guibas *On discrete killing vector fields and patterns on surfaces*, SGP 2010
Pattern Generation
Multiple Continuous Symmetries

First Eigenvector  #2  #3  #4
Pattern Generation
Conclusions

Intrinsic Symmetry Detection:

- Formulated as finding *intrinsic* distance-preserving maps.
- Often solved using isometric matching techniques.
- Theoretically equivalent to extrinsic symmetry detection but in higher dimensional space.
- Continuous symmetries treated with differential methods.

Open problems:

- Good theory for the approximate setting.
- Practical automatic methods.
- Better understanding of the correct deformation space.