STD: Student's t-Distribution of Slopes for Microfacet Based BSDFs

Supplemental material 1

Abstract

This paper focuses on microfacet reflectance models, and more precisely on the definition of a new and more general distribution function, which includes both Beckmann's and GGX distributions widely used in the computer graphics community. Therefore, our model makes use of an additional parameter γ, which controls the distribution function slope and tail height. It actually corresponds to a bivariate Student's t-distribution in slopes space and it is presented with the associated analytical formulation of the geometric attenuation factor derived from Smith representation. We also provide the analytical derivations for importance sampling isotropic and anisotropic materials. As shown in the results, this new representation offers a finer control of a wide range of materials, while extending the capabilities of fitting parameters with captured data.

This file contains the mathematical justifications concerning the Student's t Normal Distribution Function (STD). We provide all the mathematical details and links for all the expressions given in the paper: in-depth presentation of the new distributions family; proof that STD actually corresponds to Beckmann when $\gamma \to \infty$; analytical smith-Bourlier GAF derivation based on STD, and discussion about correlated and uncorrelated GAFs; formulas for the specific cases of the discrete values of γ; details about the analytical cumulative density function and its inversion for importance sampling. We also provide the demonstration for using the visible normals distribution and the constraints which should be handled in the practical case, as well as some implementation details.

Contents

1 A new family of normal distribution functions

Several normal distributions proposed in the literature define (in addition to roughness often denoted as σ) a second parameter for controlling the shape of the function [12, 2, 4, 10]. Unfortunately, to the best of our knowledge, none of them provide an analytical derivation of the Smith's GAF. We address this issue and propose a new family of normal distributions. The latter is inspired by the ABC model described by Church et al. [6], which was first introduced and adapted to the slopes distribution of microfacets by Löw et al. [12].

Our goal is to generalize/include both Beckmann and GGX distributions, since they are very popular in the computer graphics community, and they are also the only ones which offer an analytical Smith's GAF formulation. Note that the model proposed by Burley et al. [4] corresponds to a generalization of GGX only while the work proposed by Holzschuch et al. [10] corresponds to a generalization of Beckmann's distribution only.

The general expression of this new distribution family is given by:

$$D^G(\theta_m) = \frac{A}{\pi \cos^4 \theta_m \left(1 + B \tan^2 \theta_m\right)^C}. \tag{1}$$

To ensure the normalization of the distribution, A is expressed using two intermediate values B and C:

$$D^G(\theta_m) = \frac{(C-1)B}{\pi \cos^4 \theta_m \left(1 + B \tan^2 \theta_m\right)^C}, \tag{2}$$

with $B \neq 0$ and $C \neq 1$. GGX is obtained with $B = \frac{1}{\sigma^2}$ and $C = 2$. Finally, D^G tends to Beckmann when $C \to \infty$ and BC tends to $\frac{1}{\sigma^2}$.

The slopes distribution of our general formulation is:

$$P_{22}^G(p, q) = \frac{(C-1)B}{\pi \left(1 + B(p^2 + q^2)\right)^C}. \tag{3}$$

This latter equation is actually a standard bivariate Student's t-distribution. The one dimensional distribution of slopes in the incidence plane is given by:

$$P_2^G(q) = \frac{\Gamma(C - \frac{1}{2})\sqrt{B}(C-1)}{\Gamma(C)\sqrt{\pi}(1 + Bq^2)^{C-\frac{1}{2}}}. \tag{4}$$

The latter equation is computed in the same manner as for the STD distribution and explained in Section 3. Finally, the $\Lambda^G(\theta)$ function which appears in the Smith's GAF formulation is:

$$\begin{aligned}
\Lambda^G(\theta) &= \frac{1}{\mu} \int_{\mu}^{+\infty} (q - \mu) P_2^G(q)\, dq \\
&= \frac{\Gamma(C - \frac{1}{2})}{\Gamma(C)\sqrt{\pi}} \left(\frac{(C-1)(1 + B\mu^2)^{\frac{3}{2} - C}}{\mu\sqrt{B}(2C-3)} + \mu\sqrt{B}(C-1)\ {}_2F_1\left(\frac{1}{2}, C - \frac{1}{2}; \frac{3}{2}; -B\mu^2\right) \right) \\
&- \frac{1}{2},
\end{aligned} \tag{5}$$

where $\mu = \cot\theta$. The mathematical development is the same as for STD expressed in Section 3.

It is possible to determine the n^{th} moment of the one dimensional distribution of slopes $P_2^G(q)$ with

$$\nu_n = \int_{-\infty}^{+\infty} q^n P_2^G(q) dq. \quad (6)$$

Note that the moment ν_n is defined only for $C > 1 + \frac{n}{2}$. The odd order moments are zero, the distribution is standardized ($\nu_1 = 0$ for $C > \frac{3}{2}$) and symmetric ($\nu_3 = 0$ for $C > \frac{5}{2}$). For $n = 2k$ even number, we have for $C > 1 + k$:

$$\nu_{2k} = \frac{B^{-k}\Gamma(k + \frac{1}{2})\Gamma(C - 1 - k)}{\sqrt{\pi}\Gamma(C - 1)}. \quad (7)$$

The variance, defined for $C > 2$, is $\nu_2 = \frac{1}{2B(C-2)}$.

The excess kurtosis (which measures the heaviness of the distribution tail) is defined for $C > 3$. It does not depend of B and is given by $\frac{\nu_4}{\nu_2^2} - 3 = \frac{3}{C-3}$.

1.1 Interesting subsets of distributions

In the following, we denote $\gamma = C$. To be derived easily for anisotropy, a distribution must be *shape invariant*. This characteristic is obtained if $B = \frac{f}{\sigma^2}$ [1]. Subsequently, various sub-family of distributions can be derived.

With $f = 1$
The distribution is

$$D^{HC}(\theta_m) = \frac{(\gamma - 1)\sigma^{2\gamma-2}}{\pi \cos^4\theta_m(\sigma^2 + \tan^2\theta_m)^\gamma}. \quad (8)$$

and

$$\Lambda^G(\theta) = \frac{\Gamma(\gamma - \frac{1}{2})}{\Gamma(\gamma)\sqrt{\pi}} \left(\frac{\sigma}{\mu} \frac{\gamma - 1}{2\gamma - 3} (1 + \frac{\mu^2}{\sigma^2})^{\frac{3}{2}-\gamma} + \frac{\mu}{\sigma}(\gamma - 1)\ {}_2F_1\left(\frac{1}{2}, \gamma - \frac{1}{2}; \frac{3}{2}; -\frac{\mu^2}{\sigma^2}\right)\right) - \frac{1}{2},$$

which corresponds to the Hyper-Cauchy distribution first introduced by Wellems et al. [14] and used to fit measured BRDF by Butler and Marciniak [5] without Smith's GAF evaluation. The GGX distribution is included by Hyper-Cauchy (when $\gamma = 2$). However, when $\gamma \to \infty$, the distribution becomes a Dirac distribution and does not encompass the Beckmann distribution. Intuitively, when γ increases, the surface is smoother. This behavior is already controlled by the roughness σ parameter, reducing the interest of having a new parameter to control the distribution shape. In addition, the variance associated with Hyper-Cauchy strongly decreases when γ increases, which is an significant weakness for rendering purposes.

With $f = \frac{1}{(\gamma-2)}$
The distribution is:

$$D(\theta_m) = \frac{(\gamma - 1)(\gamma - 2)^{\gamma-1}\sigma^{2\gamma-2}}{\pi \cos^4\theta_m((\gamma - 2)\sigma^2 + \tan^2\theta_m)^\gamma}. \quad (9)$$

The variance of this distribution is $\frac{\sigma^2}{2}$ for $\gamma > 2$, which does not depend on γ. This family of distributions encompasses Beckmann when $\gamma \to \infty$, but it is not defined when $\gamma = 2$ and does not include GGX.

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[1] Heitz shows that a distribution is *shape invariant* if it has the form $\frac{f(\frac{\tan\theta}{\sigma})}{\sigma^2\cos^4\theta}$ [8].

With $f = \frac{1}{(\gamma-1)}$
The distribution corresponds to Student's t Normal Distribution Function (STD). Our paper and the following sections of this document detail all the expressions of the distribution, its associated Smith's GAF and importance sampling analytical representations. This solution appears as a good compromise. It includes GGX and Beckmann; its variance moderately changes with γ. A comparison is shown in Figure 1 between this STD normal distribution and the Hyper-Cauchy one.

Finally, note that $B = \frac{1}{\sigma^2(\gamma-x)}$ with x a real number offers more flexibility to define new distribution subsets. We think that our general expression of distributions opens future insights in this research field.

1.2 Anisotropy and Importance Sampling

As mentioned previously, the distribution is *shape invariant* if the expression of B has the form $\frac{f}{\sigma^2}$. Starting with this assumption of shape invariance $B = \frac{f}{\sigma^2}$, it is possible to define the anisotropic version for the general distribution:

$$D^G(m) = \frac{(C-1)f}{\pi\sigma_x\sigma_y\cos^4\theta_m\left(1 + f\ A(\varphi_m)\tan^2\theta_m\right)^C} \tag{10}$$

with $A(\varphi_m) = \left(\frac{\cos^2(\varphi_m)}{\sigma_x^2} + \frac{\sin^2(\varphi_m)}{\sigma_y^2}\right)$. Importance sampling can also be performed with this general expression. Following the same mathematical reasoning as in Section 6, with (ξ_1, ξ_2) two uniform random numbers in $[0,1)^2$, θ_m and φ_m can be importance sampled with:

$$\varphi_m = \arctan\left(\frac{\sigma_y}{\sigma_x}\tan(2\pi\xi_1)\right) \tag{11}$$

and

$$\theta_m = \arctan\sqrt{\frac{(1-\xi_2)^{\frac{1}{1-C}} - 1}{f\ A(\varphi_m)}}. \tag{12}$$

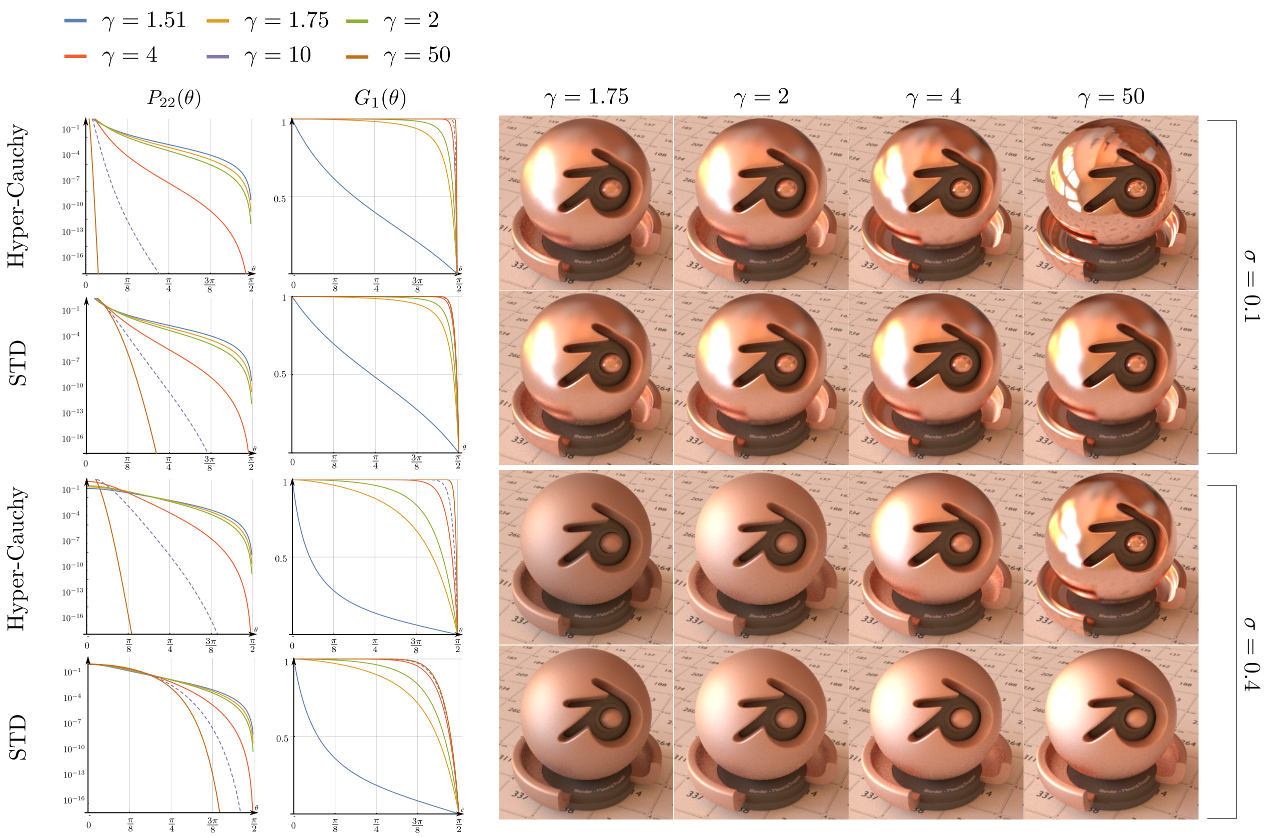

Figure 1: Comparison between STD and Hyper-Cauchy. Note that the Hyper-Cauchy tends to a dirac when the γ parameter increases, the surface becomes smoother; this behavior already corresponds to the influence of the roughness parameter σ.

2 STD = Beckmann when $\gamma \to \infty$

STD is equivalent to the Beckmann distribution when $\gamma \to \infty$. The Beckmann distribution D^{Beck} can be written as a power series:

$$D^{Beck}(\theta_m) = \frac{\exp(-\tan^2\theta_m/\sigma^2)}{\pi\sigma^2\cos^4\theta_m} = \frac{1}{\pi\sigma^2\cos^4\theta_m}\sum_{k=0}^{+\infty}\frac{1}{k!}\left(\frac{-\tan^2\theta_m}{\sigma^2}\right)^k.$$

Similarly, the STD distribution can be expanded with a Taylor series:

$$\begin{aligned}
D^{STD}(\theta_m) &= \frac{(\gamma-1)^\gamma\sigma^{2\gamma-2}}{\pi\cos^4\theta_m\left((\gamma-1)\sigma^2+\tan^2\theta_m\right)^\gamma} = \frac{1}{\pi\cos^4\theta_m\sigma^2\left(1+\frac{\tan^2\theta_m}{(\gamma-1)\sigma^2}\right)^\gamma} \\
&= \frac{1}{\pi\sigma^2\cos^4\theta_m}\left(\sum_{k=0}^{+\infty}\binom{-\gamma}{k}\left(\frac{\tan^2\theta_m}{(\gamma-1)\sigma^2}\right)^k\right) \\
&= \frac{1}{\pi\sigma^2\cos^4\theta_m}\left(\sum_{k=0}^{+\infty}\frac{\gamma(\gamma+1)\cdots(\gamma+k-1)}{(\gamma-1)^k k!}\left(\frac{-\tan^2\theta_m}{\sigma^2}\right)^k\right),
\end{aligned}$$

with $\lim\limits_{\gamma\to+\infty}\frac{\gamma(\gamma+1)\cdots(\gamma+k-1)}{(\gamma-1)^k} = 1$, and therefore:

$$\lim_{\gamma\to+\infty} D^{STD}(\theta_m) = \frac{1}{\pi\sigma^2\cos^4\theta_m}\sum_{k=0}^{+\infty}\frac{1}{k!}\left(\frac{-\tan^2\theta_m}{\sigma^2}\right)^k = D^{Beck}(\theta_m).$$

3 GAF computation for STD

3.1 Uncorrelated GAF

The GAF is computed using the same process as previously proposed in [3, 8, 13]. The normal distribution is first expressed in the slopes space:

$$P_{22}^{STD}(p,q) = D^{STD}(\theta_m) * \cos^4\theta_m = \frac{(\gamma-1)^{\gamma}\sigma^{2\gamma-2}}{\pi((\gamma-1)\sigma^2+p^2+q^2)^{\gamma}}, \tag{13}$$

with $p^2+q^2=\tan^2\theta_m$. The one dimensional distribution of slopes in the incidence plane is given by:

$$\begin{aligned} P_2^{STD}(q) &= \int_{-\infty}^{\infty} P_{22}^{STD}(p,q)\ dp \\ &= \frac{(\gamma-1)^{\gamma}\sigma^{2\gamma-2}}{\pi\left((\gamma-1)\sigma^2+q^2\right)^{\gamma}} \int_{-\infty}^{\infty} \frac{dp}{\left(\frac{p^2}{(\gamma-1)\sigma^2+q^2}+1\right)^{\gamma}}. \end{aligned} \tag{14}$$

Using the substitution $t=\frac{p}{\sqrt{(\gamma-1)\sigma^2+q^2}}$, the previous integral becomes:

$$\begin{aligned} P_2^{STD}(q) &= \frac{(\gamma-1)^{\gamma}\sigma^{2\gamma-2}}{\pi\left((\gamma-1)\sigma^2+q^2\right)^{\gamma-\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{dt}{(t^2+1)^{\gamma}} \\ &= \frac{(\gamma-1)^{\gamma}\sigma^{2\gamma-2}\Gamma(\gamma-\frac{1}{2})}{\sqrt{\pi}((\gamma-1)\sigma^2+q^2)^{\gamma-\frac{1}{2}}\Gamma(\gamma)}, \end{aligned} \tag{15}$$

which is valid for $\gamma > 1/2$ and with $\Gamma(\gamma)=\int_0^{+\infty} x^{\gamma-1}e^{-x}\ dx$ the Gamma function. Finally, the GAF $G_1(\theta)$ is defined as:

$$G_1^{STD}(\theta) = \frac{1}{1+\Lambda^{STD}(\mu)}, \tag{16}$$

where $\mu=\cot\theta$ and

$$\begin{aligned} \Lambda^{STD}(\theta) &= \frac{1}{\mu}\int_{\mu}^{+\infty}(q-\mu)P_2^{STD}(q)\ dq \\ &= \frac{(\gamma-1)^{\gamma}\sigma^{2\gamma-2}}{\sqrt{\pi}\mu}\frac{\Gamma(\gamma-\frac{1}{2})}{\Gamma(\gamma)}\int_{\mu}^{+\infty}(q-\mu)\frac{dq}{((\gamma-1)\sigma^2+q^2)^{\gamma-\frac{1}{2}}} \\ &= \frac{(\gamma-1)^{\gamma}\sigma^{2\gamma-2}}{\sqrt{\pi}\mu}\frac{\Gamma(\gamma-\frac{1}{2})}{\Gamma(\gamma)}\left(I_1(\theta)-I_2(\theta)\right), \end{aligned} \tag{17}$$

where

$$\begin{aligned} I_1(\theta) &= \int_{\mu}^{+\infty}\frac{q}{((\gamma-1)\sigma^2+q^2)^{\gamma-1/2}}\ dq \\ &= -\frac{\left((\gamma-1)\sigma^2+\mu^2\right)^{3/2-\gamma}}{3-2\gamma}, \end{aligned} \tag{18}$$

and

$$\begin{aligned} I_2(\theta) &= \mu\int_{\mu}^{+\infty}\frac{dq}{((\gamma-1)\sigma^2+q^2)^{\gamma-\frac{1}{2}}} \\ &= \frac{\mu}{(\gamma-1)^{\gamma-\frac{1}{2}}\sigma^{2\gamma-1}}\int_{\mu}^{+\infty}\frac{dq}{\left(1+\frac{q^2}{(\gamma-1)\sigma^2}\right)^{\gamma-\frac{1}{2}}}. \end{aligned} \tag{19}$$

With the substitution $t = \frac{q}{\sqrt{(\gamma-1)\sigma^2}}$, the integral I_2 becomes:

$$\begin{aligned} I_2(\theta) &= \frac{\mu\sqrt{(\gamma-1)\sigma^2}}{(\gamma-1)^{\gamma-\frac{1}{2}}\sigma^{2\gamma-1}} \int_{\frac{\mu}{\sqrt{(\gamma-1)\sigma^2}}}^{+\infty} \frac{dt}{(1+t^2)^{\gamma-\frac{1}{2}}} \\ &= \frac{\mu}{((\gamma-1)\sigma^2)^{\gamma-1}} \left(\frac{\sqrt{\pi}\,\Gamma(\gamma-1)}{2\,\Gamma(\gamma-\frac{1}{2})} - \frac{\mu\ {}_2F_1\left(\frac{1}{2},\gamma-\frac{1}{2};\frac{3}{2};-\frac{\mu^2}{(\gamma-1)\sigma^2}\right)}{\sqrt{(\gamma-1)\sigma^2}} \right), \end{aligned} \tag{20}$$

where ${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{+\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}$ is the Gauss hypergeometric function. Using Equations 19 and 20 in Equation 17, Λ^{STD} can be written as follows:

$$\begin{aligned} \Lambda^{STD}(\theta) &= \frac{\Gamma(\gamma-\frac{1}{2})}{\Gamma(\gamma)\sqrt{\pi}} \left(\frac{(\gamma-1)^{\gamma}}{2\gamma-3} \frac{\sigma((\gamma-1)+\frac{\mu^2}{\sigma^2})^{\frac{3}{2}-\gamma}}{\mu} + \sqrt{\gamma-1}\frac{\mu}{\sigma} \times {}_2F_1\left(\frac{1}{2},\gamma-\frac{1}{2};\frac{3}{2};\frac{-\mu^2}{(\gamma-1)\sigma^2}\right)\right) \\ &- \frac{1}{2}. \end{aligned} \tag{21}$$

3.2 Correlated GAF

For comparison purposes with GGX and Beckmann distributions already implemented in Mitsuba [11], all the renderings in the paper are performed with the uncorrelated Smith's GAF:

$$G(i,o,m) = G_1(i,m) \times G_1(o,m) = \frac{1}{1+\Lambda(i)} \times \frac{1}{1+\Lambda(o)}. \tag{22}$$

The STD distribution can also obviously be used with the correlated GAF:

$$G(i,o,m) = \frac{1}{1+\Lambda(i)+\Lambda(o)}, \tag{23}$$

without any change in all the mathematical developments and quality results. Note that $G = 0$ if $i \cdot m < 0$ or $o \cdot m < 0$. The correlated GAF is considered as physically more plausible [8] and limits surface darkening (without removing it) for high roughness values as shown in Figure 2. Although the correlated GAF is known to be preferable for lowering masking-shadowing effects, darkening still remains important because light interreflections between microfacets are not handled.

As the uncorrelated GAF impacts the surface brightness, the stability of the STD distribution could be questionned for low value of γ (because visible normals importance sampling is not used). We have made several tests with the correlated GAF, and when $\gamma < 1.8$ without using visible normals (i.e. the worst case) the noise is actually slightly more visible (Figure 3). Note that in both cases, some spikes remain visible and importance sampling using the distribution of visible normals should remove (or at least reduce) this effect. This point is discussed in Section 7.

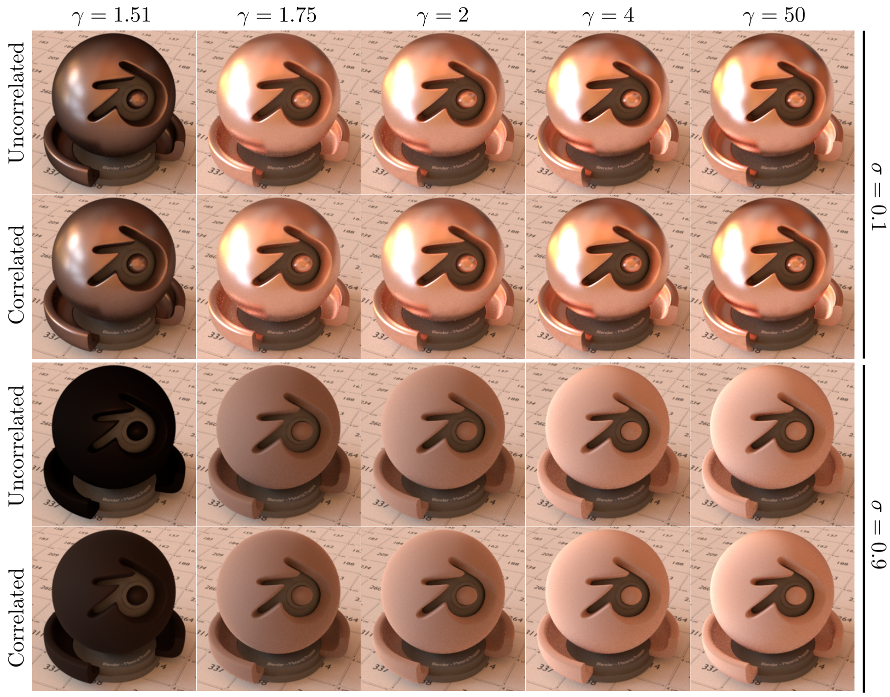

Figure 2: The correlated GAF is physically more plausible and limits the darkening issue due to the energy loss. This problem still remains important when the roughness σ is large and the γ parameter is less than 2. All the renderings are made with the STD normals distribution.

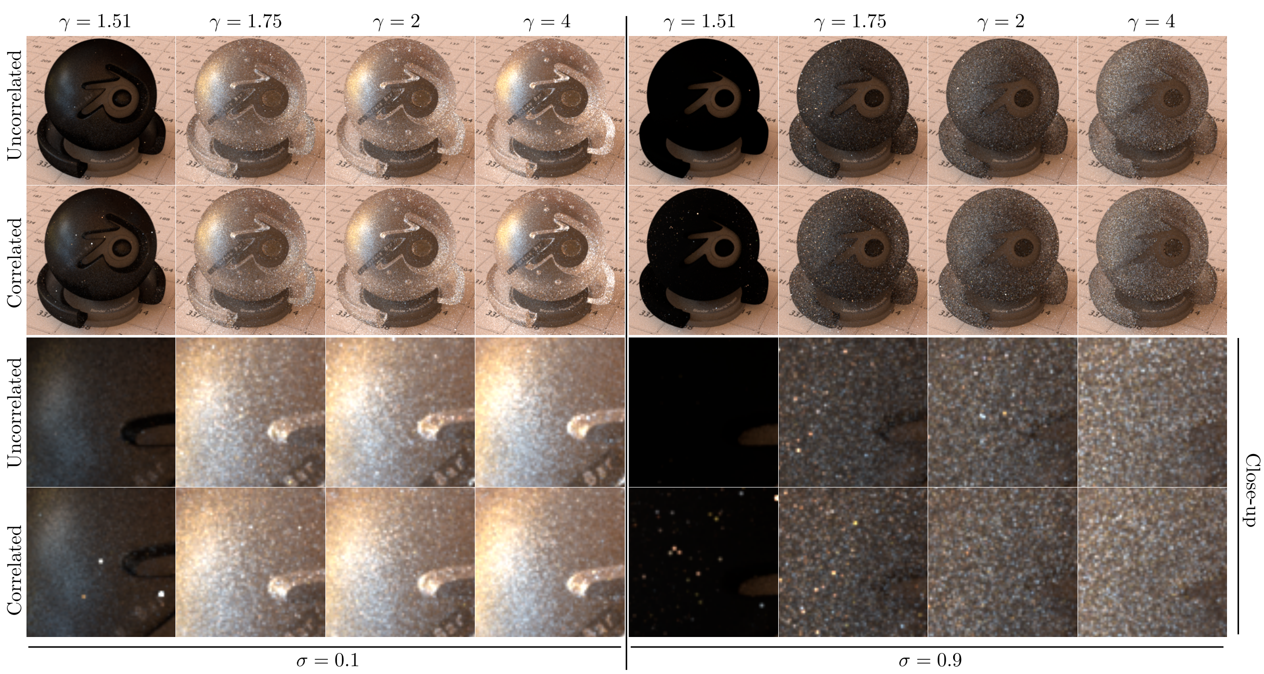

Figure 3: Comparison of importance sampling strategies (64 samples per pixel) on rough dielectric surface with a correlated and uncorrelated GAF. Some spikes are visible in both cases but when $\gamma < 1.8$, this phenomenon is slightly more visible with the correlated GAF.

4 Discrete formulas for the STD GAF

Special functions appearing in the GAF and more particularly the Gauss hypergeometric function ${}_2F_1$ can impact negatively the computation time. It is possible to derive analytic equations for particular values of γ. The mathematical process remains the same as for the general cases (see Section 3), but the derivations of the one dimensional distribution of slopes P_2^{STD} and the Λ^{STD} function differ slightly:

$$P_2^{STD}(q) = \frac{(\gamma-1)^\gamma \sigma^{2\gamma-2}}{\pi\left((\gamma-1)\sigma^2+q^2\right)^{\gamma-\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{dt}{(t^2+1)^\gamma},$$

and using $t = \tan\ x$:

$$\begin{aligned} P_2^{STD}(q) &= \frac{(\gamma-1)^\gamma \sigma^{2\gamma-2}}{\pi\left((\gamma-1)\sigma^2+q^2\right)^{\gamma-\frac{1}{2}}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{(\tan^2 x+1)^{\gamma-1}} \\ &= F \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2 x)^{\gamma-1}\ dx, \end{aligned} \tag{24}$$

with $F = \frac{(\gamma-1)^\gamma \sigma^{2\gamma-2}}{\pi((\gamma-1)\sigma^2+q^2)^{\gamma-\frac{1}{2}}}$. The integral can be solved with the integer and half-integer values for γ. In these cases, the infinite sum inside the ${}_2F_1$ function is replaced by a finite one, as shown below.

4.1 Integer discrete case

Let $\mathbb{N}^{+1} = \{\mathbb{N} \mid \gamma > 1\}$ denoting the positive integer set. Using the Euler's formula $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and the Newton generalized binomial theorem, Equation 24 becomes:

$$\begin{aligned} P_2^{STD}(q) &= \frac{F}{2^{2\gamma-2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^{ix}+e^{-ix})^{2\gamma-2}\ dx = \frac{F}{2^{2\gamma-2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_{k=0}^{2\gamma-2} \binom{2\gamma-2}{k} e^{i(2\gamma-2-2k)x}\ dx \\ &= \frac{F}{2^{2\gamma-2}} \sum_{k=0}^{2\gamma-2} \binom{2\gamma-2}{k} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(2\gamma-2-2k)x}\ dx \\ &= \frac{F}{2^{2\gamma-2}} \sum_{k=0}^{2\gamma-2} \frac{\binom{2\gamma-2}{k}}{i(2\gamma-2-2k)} \left(e^{i(2\gamma-2-2k)\frac{\pi}{2}} - e^{-i(2\gamma-2-2k)\frac{\pi}{2}}\right) \\ &= \frac{F}{2^{2\gamma-2}} \sum_{k=0}^{2\gamma-2} \frac{\binom{2\gamma-2}{k}}{(\gamma-1-k)} \sin\left((\gamma-1-k)\pi\right). \end{aligned} \tag{25}$$

In Equation 25, the all terms of the sum are equal to zero except when $k = \gamma - 1$ for which we have:

$$\begin{aligned} P_2^{STD}(q) &= \frac{F}{2^{2\gamma-2}} \binom{2\gamma-2}{\gamma-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^0\ dx = \frac{F}{2^{2\gamma-2}} \binom{2\gamma-2}{\gamma-1} \pi \\ &= \frac{\pi F}{2^{2\gamma-2}} \frac{(2\gamma-2)!}{(\gamma-1)!^2}, \end{aligned}$$

and finally with some mathematical simplifications:

$$P_2^{STD}(q) = \frac{(\gamma-1)^\gamma \sigma^{2\gamma-2}}{2^{2\gamma-2}\left((\gamma-1)\sigma^2+q^2\right)^{\gamma-\frac{1}{2}}} \times \frac{(2\gamma-2)!}{(\gamma-1)!^2}. \tag{26}$$

Equation 26 is used to evaluate the Λ^{STD} function:

$$\begin{aligned}
\Lambda^{STD}(\theta) &= \frac{1}{\mu}\int_{\mu}^{+\infty}(q-\mu)P_2^{STD}(q)\ dq \\
&= \frac{(\gamma-1)^{\gamma}\sigma^{2\gamma-2}(2\gamma-2)!}{2^{2\gamma-2}(\gamma-1)!^2\mu}\left(\int_{\mu}^{+\infty}\frac{q\ dq}{((\gamma-1)\sigma^2+q^2)^{\gamma-\frac{1}{2}}}-\mu\int_{\mu}^{+\infty}\frac{dq}{((\gamma-1)\sigma^2+q^2)^{\gamma-\frac{1}{2}}}\right) \\
&= \frac{(\gamma-1)^{\gamma}\sigma^{2\gamma-2}(2\gamma-2)!}{2^{2\gamma-2}(\gamma-1)!^2\mu}\left(\frac{\left((\gamma-1)\sigma^2+\mu^2\right)^{\frac{3}{2}-\gamma}}{2\gamma-3}-\right. \\
&\left.\frac{\mu}{(\gamma-1)^{\gamma-\frac{1}{2}}\sigma^{2\gamma-1}}\int_{\mu}^{+\infty}\frac{dq}{\left(\left(\frac{q}{\sqrt{\gamma-1}\sigma^2}\right)+1\right)^{\gamma-\frac{1}{2}}}\right),
\end{aligned}$$

using $t=\frac{\mu}{\sigma\sqrt{\gamma-1}}$ in the integral provides:

$$\Lambda^{STD}(\theta) = \frac{(\gamma-1)^{\gamma}\sigma^{2\gamma-2}(2\gamma-2)!}{2^{2\gamma-2}(\gamma-1)!^2\mu}\left(\frac{\left((\gamma-1)\sigma^2+\mu^2\right)^{\frac{3}{2}-\gamma}}{2\gamma-3}-\frac{\mu}{((\gamma-1)\sigma^2)^{\gamma-1})}\int_{\frac{\mu}{\sqrt{\gamma-1}\sigma}}^{+\infty}\frac{dt}{(t^2+1)^{\gamma-\frac{1}{2}}}\right). \tag{27}$$

With a second substitution $t=\tan x$, the latter becomes:

$$\begin{aligned}
\int_{\frac{\mu}{\sqrt{\gamma-1}\sigma}}^{+\infty}\frac{dt}{(t^2+1)^{\gamma-\frac{1}{2}}} &= \int_{\arctan\left(\frac{\mu}{\sqrt{\gamma-1}\sigma}\right)}^{\frac{\pi}{2}}(\cos x)^{2\gamma-3}\ dx \\
&= \int_{\arctan\left(\frac{\mu}{\sqrt{\gamma-1}\sigma}\right)}^{\frac{\pi}{2}}\cos x\ (1-\sin^2 x)^{\gamma-2}\ dx \\
&= \int_{\arctan\left(\frac{\mu}{\sqrt{\gamma-1}\sigma}\right)}^{\frac{\pi}{2}}\cos x\sum_{k=0}^{\gamma-2}\binom{\gamma-2}{k}(-\sin^2 x)^k\ dx \\
&= \sum_{k=0}^{\gamma-2}\binom{\gamma-2}{k}\frac{(-1)^k}{2k+1}\left(1-\sin^{2k+1}\left(\arctan\frac{\mu}{\sqrt{\gamma-1}\sigma}\right)\right) \\
&= \sum_{k=0}^{\gamma-2}\binom{\gamma-2}{k}\frac{(-1)^k}{2k+1}\left(1-\left(\frac{\mu}{\sigma\sqrt{(\gamma-1)+\frac{\mu^2}{\sigma^2}}}\right)^{2k+1}\right).
\end{aligned} \tag{28}$$

Using Equation 28 in 27 and re-ordering terms:

$$\begin{aligned}
\Lambda^{STD}(\theta) &= (\gamma-1)A_1 \\
&\left(\frac{\sigma}{\mu}\sqrt{\gamma-1}\frac{(1+\frac{\mu^2}{(\gamma-1)\sigma^2})^{3/2-\gamma}}{2\gamma-3}-A_2\right),
\end{aligned} \tag{29}$$

where

$$\tag{30}$$

$$
\begin{aligned}
A_1 &= \frac{(2\gamma-2)!}{2^{2\gamma-2}(\gamma-1)!^2}, \\
A_2 &= \sum_{k=0}^{\gamma-2} \binom{\gamma-2}{k} \frac{(-1)^k}{2k+1} A_3(k), \\
A_3(k) &= 1 - \left(\frac{\mu}{\sigma\sqrt{(\gamma-1)+\frac{\mu^2}{\sigma^2}}} \right)^{2k+1}.
\end{aligned}
$$

4.2 Half-integer discrete case

Let $\mathbb{N}^{1/2} = \{n+1/2 \mid n \in \mathbb{N} \ \& \ n \geq 2\}$ denoting the positive half-integer set, Equation 24 becomes:

$$
P_2^{STD}(q) = F \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2 x)^{A-\frac{1}{2}} \ dx,
$$

with $\gamma = A + \frac{1}{2}$. Thus:

$$
\begin{aligned}
P_2^{STD}(q) &= F \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{2A-1} \ dx, \qquad \text{where } A-1 \text{ is a natural number}, \\
&= F \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x (1-\sin^2 x)^{A-1} \ dx, \\
&= F \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \sum_{k=0}^{A-1} \binom{A-1}{k} (-\sin^2 x)^k \ dx \\
&= F \sum_{k=0}^{\gamma-\frac{3}{2}} \binom{\gamma-\frac{3}{2}}{k} (-1)^k \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x (\sin x)^{2k} \ dx \\
&= F \sum_{k=0}^{\gamma-\frac{3}{2}} \binom{\gamma-\frac{3}{2}}{k} \frac{(-1)^k 2}{2k+1} \\
&= F \sum_{k=0}^{\gamma-\frac{3}{2}} \binom{\gamma-\frac{3}{2}}{k} \frac{(-1)^k 2}{2k+1} \\
&= \frac{2(\gamma-1)^\gamma \sigma^{2\gamma-2}}{\pi\left((\gamma-1)\sigma^2+q^2\right)^{\gamma-\frac{1}{2}}} \sum_{k=0}^{\gamma-\frac{3}{2}} \binom{\gamma-\frac{3}{2}}{k} \frac{(-1)^k}{2k+1}. \qquad (31)
\end{aligned}
$$

Equation 31 is used to evaluate the Λ^{STD} function:

$$\begin{aligned}\Lambda^{STD}(\theta) &= \frac{2(\gamma-1)^{\gamma}\sigma^{2\gamma-2}}{\pi\left((\gamma-1)\sigma^2+q^2\right)^{\gamma-\frac{1}{2}}}\sum_{k=0}^{\gamma-\frac{3}{2}}\binom{\gamma-\frac{3}{2}}{k}(-1)^k\left(\int_{\mu}^{+\infty}\frac{q\ dq}{\left((\gamma-1)\sigma^2+q^2\right)^{\gamma-\frac{1}{2}}}-\right.\\ &\left.\mu\int_{\mu}^{+\infty}\frac{dq}{\left((\gamma-1)\sigma^2+q^2\right)^{\gamma-\frac{1}{2}}}\right),\end{aligned} \tag{32}$$

and using the same process as for Equation 27:

$$\begin{aligned}\Lambda^{STD}(\theta) &= \frac{2(\gamma-1)^{\gamma}\sigma^{2\gamma-2}}{\pi\left((\gamma-1)\sigma^2+q^2\right)^{\gamma-\frac{1}{2}}}\sum_{k=0}^{\gamma-\frac{3}{2}}\binom{\gamma-\frac{3}{2}}{k}(-1)^k\left(\frac{\left((\gamma-1)\sigma^2+\mu^2\right)^{\frac{3}{2}-\gamma}}{2\gamma-3}-\right.\\ &\left.\frac{\mu}{\left((\gamma-1)\sigma^2\right)^{\gamma-1)}}\int_{\frac{\mu}{\sqrt{\gamma-1}\sigma}}^{+\infty}\frac{dt}{(t^2+1)^{\gamma-\frac{1}{2}}}\right).\end{aligned} \tag{33}$$

With a second substitution $t=\tan x$, the latter integral becomes:

$$\begin{aligned}\int_{\frac{\mu}{\sqrt{\gamma-1}\sigma}}^{+\infty}\frac{dt}{(t^2+1)^{\gamma-\frac{1}{2}}} &= \int_{\arctan\left(\frac{\mu}{\sqrt{\gamma-1}\sigma}\right)}^{\frac{\pi}{2}}(\cos x)^{2\gamma-3}\ dx\\ &= \int_{\arctan\left(\frac{\mu}{\sqrt{\gamma-1}\sigma}\right)}^{\frac{\pi}{2}}\left(\frac{e^{ix}+e^{-ix}}{2}\right)^{2\gamma-3}\ dx\\ &= \frac{1}{2^{2\gamma-3}}\int_{\arctan\left(\frac{\mu}{\sqrt{\gamma-1}\sigma}\right)}^{\frac{\pi}{2}}\left(e^{ix}+e^{-ix}\right)^{2\gamma-3}\ dx\\ &= \frac{1}{2^{2\gamma-3}}\int_{\arctan\left(\frac{\mu}{\sqrt{\gamma-1}\sigma}\right)}^{\frac{\pi}{2}}\sum_{k=0}^{2\gamma-3}\binom{2\gamma-3}{k}e^{i(2\gamma-3-k)x}e^{-ikx}\ dx\\ &= \frac{1}{2^{2\gamma-3}}\int_{\arctan\left(\frac{\mu}{\sqrt{\gamma-1}\sigma}\right)}^{\frac{\pi}{2}}\sum_{k=0}^{2\gamma-3}\binom{2\gamma-3}{k}e^{i(2\gamma-3-2k)x}\ dx.\end{aligned} \tag{34}$$

The series over the exponential can be re-organized. Indeed, the first term plus the last one corresponds to the Euler formula of the cosine, the second one plus the second last one also and so on. As the series is over an odd number of term, the remaining is equal to $e^0=1$. Finally:

$$\begin{aligned}\int_{\frac{\mu}{\sqrt{\gamma-1}\sigma}}^{+\infty}\frac{dt}{(t^2+1)^{\gamma-\frac{1}{2}}} &= \frac{1}{2^{2\gamma-4}}\sum_{k=0}^{\gamma-\frac{5}{2}}\binom{2\gamma-3}{k}\int_{\arctan\left(\frac{\mu}{\sqrt{\gamma-1}\sigma}\right)}^{\frac{\pi}{2}}\cos\left((2\gamma-3-2k)x\right)\ dx+\\ &\quad\frac{1}{2^{2\gamma-3}}\binom{2\gamma-3}{k}\left(\frac{\pi}{2}-\arctan\left(\frac{\mu}{\sqrt{\gamma-1}\sigma}\right)\right)\\ &= \frac{1}{2^{2\gamma-4}}\sum_{k=0}^{\gamma-\frac{5}{2}}\binom{2\gamma-3}{k}\frac{\left(-\sin\left((2\gamma-3-2k)\arctan\left(\frac{\mu}{\sqrt{(\gamma-1)}\sigma}\right)\right)\right)}{2\gamma-3-2k}+\\ &\quad\frac{1}{2^{2\gamma-3}}\binom{2\gamma-3}{\gamma-3/2}\left(\frac{\pi}{2}-\arctan\left(\frac{\mu}{\sqrt{(\gamma-1)}\sigma}\right)\right)\end{aligned} \tag{35}$$

The complete Λ^{STD} for the half-integer values of γ can be summarized using Equation 29 with:

$$A_1 = \frac{2}{\pi} \sum_{k=0}^{\gamma-3/2} \binom{\gamma - 3/2}{k} \frac{(-1)^k}{2k+1}, \tag{36}$$

and

$$A_2 = \frac{1}{2^{2\gamma-4}} A_{21} + \frac{1}{2^{2\gamma-3}} A_{22}, \tag{37}$$

and finally:

$$\begin{aligned} A_{21} &= \sum_{k=0}^{\gamma-5/2} \binom{2\gamma-3}{k} \frac{1}{2\gamma-3-2k} \times \\ &\quad \left(-\sin\left((2\gamma-3-2k)\arctan\left(\frac{\mu}{\sqrt{(\gamma-1)\sigma}}\right)\right)\right) \\ A_{22} &= \binom{2\gamma-3}{\gamma-3/2} \left(\frac{\pi}{2} - \arctan\left(\frac{\mu}{\sqrt{(\gamma-1)\sigma}}\right)\right) \end{aligned}$$

5 STD expressions for particular γ values

Table 1 presents some expressions for particular values of γ. This can be useful if just one of these formulation is needed. The Λ functions for anisotropic materials can be obtained using $\frac{\sigma}{\mu} = \tan\theta\sqrt{\sigma_x^2\cos^2\varphi + \sigma_y^2\sin^2\varphi}$ and the GAF is $G_1(\theta) = \frac{1}{1+\Lambda(\theta)}$.

Table 1: STD expressions for particular γ values.

$\gamma = 2$	$D^{GGX}(\theta)$ isotropic	$\Rightarrow$	$\frac{1}{\pi\sigma^2\cos^4\theta(1+\frac{\tan^2\theta}{\sigma^2})^2}$
	$D^{GGX}(\theta)$ anisotropic	$\Rightarrow$	$\frac{1}{\pi\sigma_x\sigma_y\cos^4\theta\left(1+\tan^2\theta(\frac{\cos^2\varphi}{\sigma_x^2}+\frac{\sin^2\varphi}{\sigma_y^2})\right)^2}$
	$\Lambda(\theta)$ isotropic	$\Rightarrow$	$\frac{1}{2}\left(\frac{\sqrt{\mu^2+\sigma^2}}{\mu}-1\right)$
$\gamma = 5/2$	$D^{STD}(\theta)$ isotropic	$\Rightarrow$	$\frac{1}{\pi\sigma^2\cos^4\theta(1+\frac{2\tan^2\theta}{3\sigma^2})^{5/2}}$
	$D^{STD}(\theta)$ anisotropic	$\Rightarrow$	$\frac{1}{\pi\sigma_x\sigma_y\cos^4\theta\left(1+\frac{2}{3}\tan^2\theta(\frac{\cos^2\varphi}{\sigma_x^2}+\frac{\sin^2\varphi}{\sigma_y^2})\right)^{5/2}}$
	$\Lambda(\theta)$ isotropic	$\Rightarrow$	$\frac{1}{\pi}\arctan\left(\sqrt{\frac{2}{3}}\frac{\mu}{\sigma}\right)-\frac{1}{2}+\sqrt{\frac{3}{2}}\frac{\sigma}{\pi\mu}$
$\gamma = 3$	$D^{STD}(\theta)$ isotropic	$\Rightarrow$	$\frac{8}{\pi\sigma^2\cos^4\theta(2+\frac{\tan^2\theta}{\sigma^2})^3}$
	$D^{STD}(\theta)$ anisotropic	$\Rightarrow$	$\frac{8}{\pi\sigma_x\sigma_y\cos^4\theta\left(2+\tan^2\theta(\frac{\cos^2\varphi}{\sigma_x^2}+\frac{\sin^2\varphi}{\sigma_y^2})\right)^3}$
	$\Lambda(\theta)$ isotropic	$\Rightarrow$	$\frac{1}{2}\left(\frac{\mu^4+3\mu^2\sigma^2+2\sigma^4}{\mu(\mu^2+2\sigma^2)^{3/2}}-1\right)$
$\gamma = 7/2$	$D^{STD}(\theta)$ isotropic	$\Rightarrow$	$\frac{1}{\pi\sigma^2\cos^4\theta(1+\frac{2\tan^2\theta}{5\sigma^2})^{7/2}}$
	$D^{STD}(\theta)$ anisotropic	$\Rightarrow$	$\frac{1}{\pi\sigma_x\sigma_y\cos^4\theta\left(1+\frac{2}{5}\tan^2\theta(\frac{\cos^2\varphi}{\sigma_x^2}+\frac{\sin^2\varphi}{\sigma_y^2})\right)^{7/2}}$
	$\Lambda(\theta)$ isotropic	$\Rightarrow$	$\frac{1}{3\pi}\arctan\left(\sqrt{\frac{2}{5}}\frac{\mu}{\sigma}\right)-1+5\sqrt{\frac{5}{2}}\frac{-6\mu^4+17\mu^2\sigma^2+8\sigma^4}{6\pi\mu\sigma(2\mu^2+5\sigma^2)}$
$\gamma = 4$	$D^{STD}(\theta)$ isotropic	$\Rightarrow$	$\frac{81}{\pi\sigma^2\cos^4\theta(3+\frac{\tan^2\theta}{\sigma^2})^4}$
	$D^{STD}(\theta)$ anisotropic	$\Rightarrow$	$\frac{81}{\pi\sigma_x\sigma_y\cos^4\theta\left(3+\tan^2\theta(\frac{\cos^2\varphi}{\sigma_x^2}+\frac{\sin^2\varphi}{\sigma_y^2})\right)^4}$
	$\Lambda(\theta)$ isotropic	$\Rightarrow$	$\frac{1}{2}\left(\frac{8\mu^6+60\mu^4\sigma^2+135\mu^2\sigma^4+81\sigma^6}{8\mu(\mu^2+3\sigma^2)^{5/2}}-1\right)$
$\gamma \longrightarrow \infty$	$D^{Beckmann}(\theta)$ isotropic	$\Rightarrow$	$\frac{e^{-\tan^2\theta/\sigma^2}}{\pi\sigma^2\cos^4\theta}$
	$D^{Beckmann}(\theta)$ anisotropic	$\Rightarrow$	$\frac{e^{-\tan^2\left(\frac{\cos^2\varphi}{\sigma_x^2}+\frac{\sin^2\varphi}{\sigma_y^2}\right)}}{\pi\sigma_x\sigma_y\cos^4\theta}$
	$\Lambda(\theta)$ isotropic	$\Rightarrow$	$\frac{1}{2}\left(\frac{\sigma e^{-\mu^2/\sigma^2}}{\mu\sqrt{\pi}}+\mathrm{erf}(\mu/\sigma)-1\right)$

6 Importance Sampling

The micro-facet normal m is sampled according to the probability density function $pdf^{STD}(m) = D^{STD}(m)|m \cdot n|$ and its associated cumulative distribution function:

$$\begin{aligned} cdf^{STD}(m) &= \int_{\varphi=0}^{\varphi_m}\int_{\theta=0}^{\theta_m} D^{STD}(m)\cos(\theta)\sin(\theta)\; d\theta\; d\varphi \\ &= \int_{\varphi=0}^{\varphi_m}\int_{\theta=0}^{\theta_m} \frac{\tan(\theta)}{\pi\sigma_x\sigma_y\cos^2(\theta)\left(1+A(\varphi)\tan^2(\theta)\right)^{\gamma}}\; d\theta\; d\varphi, \end{aligned} \tag{38}$$

where $A(\varphi) = \left(\frac{\cos^2(\varphi)}{\sigma_x^2} + \frac{\sin^2(\varphi)}{\sigma_y^2}\right)/(\gamma-1)$. Note that we use here the mathematical expression corresponding to the anisotropic version of STD. We first sample the micro-facet azimuthal angle φ_m; the elevation angle θ_m is sampled secondly. The choice of φ_m is independent of θ_m. Thus, the *cdf* becomes:

$$\begin{aligned} cdf^{STD}(\varphi_m) &= \int_{\varphi=0}^{\varphi_m} \frac{1}{2\pi\sigma_x\sigma_y A(\varphi)}\int_{\theta=0}^{\theta_m} \frac{2A(\varphi)\tan(\theta)}{\cos^2(\theta)}\left(1+A(\varphi)\tan^2(\theta)\right)^{-\gamma}\; d\theta\; d\varphi \\ &= \int_{\varphi=0}^{\varphi_m} \frac{1}{2\pi\sigma_x\sigma_y A(\varphi)}\frac{1}{\gamma-1} d\varphi \\ &= \int_{\varphi=0}^{\varphi_m} \frac{1}{2\pi\sigma_x\sigma_y\left(\frac{\cos^2(\varphi)}{\sigma_x^2}+\frac{\sin^2(\varphi)}{\sigma_y^2}\right)} d\varphi \\ &= \frac{\arctan\left(\frac{\sigma_x}{\sigma_y}\tan(\varphi_m)\right)}{2\pi}. \end{aligned} \tag{39}$$

The latter can be inverted (ξ_1 is a uniform random number in $[0,1)$):

$$\begin{aligned} \xi_1 &= \frac{\arctan\left(\frac{\sigma_x}{\sigma_y}\tan(\varphi_m)\right)}{2\pi} \\ &\Rightarrow \arctan\left(\frac{\sigma_x}{\sigma_y}\tan(\varphi_m)\right) = 2\pi\xi_1 \\ &\Rightarrow \varphi_m = \arctan\left(\frac{\sigma_y}{\sigma_x}\tan(2\pi\xi_1)\right). \end{aligned} \tag{40}$$

We obtain here the same result as for *GGX* and Beckmann distributions.

Knowing the azimuthal angle φ_m, the *cdf* to sample θ_m is:

$$\begin{aligned} cdf^{STD}(\theta_m) &= 2\pi\int_{\theta=0}^{\theta_m} \frac{\tan(\theta)}{\pi\sigma_x\sigma_y\cos^2(\theta)\left(1+A(\varphi_m)\tan^2(\theta)\right)^{\gamma}}\; d\theta \\ &= \frac{1}{\sigma_x\sigma_y A(\varphi_m)}\int_{\theta=0}^{\theta_m} \frac{2A(\varphi_m)\tan(\theta)}{\cos^2(\theta)}\left(1+A(\varphi_m)\tan^2(\theta)\right)^{-\gamma}\; d\theta \\ &= \frac{1}{\sigma_x\sigma_y A(\varphi_m)(1-\gamma)}\left((1+A(\varphi_m)\tan^2\theta)^{1-\gamma}-1\right) \\ &= \frac{1}{\sigma_x\sigma_y\left(\frac{\cos^2(\varphi_m)}{\sigma_x^2}+\frac{\sin^2(\varphi_m)}{\sigma_y^2}\right)}\left(1-(1+A(\varphi_m)\tan^2\theta)^{1-\gamma}\right). \end{aligned} \tag{41}$$

This *cdf* can be inverted with ξ_2 an other uniform random number in $[0,1)$:

$$
\begin{aligned}
\frac{\xi_2}{\sigma_x\sigma_y\left(\frac{\cos^2(\varphi_m)}{\sigma_x^2}+\frac{\sin^2(\varphi_m)}{\sigma_y^2}\right)} &= cdf^{STD}(\theta_m) \\
\Rightarrow \qquad \xi_2 &= \left(1-(1+A(\varphi_m)\tan^2\theta)^{1-\gamma}\right) \\
\Rightarrow \qquad \tan^2\theta_m &= \frac{(1-\xi_2)^{\frac{1}{1-\gamma}}-1}{A(\varphi_m)} \\
\Rightarrow \qquad \theta_m &= \arctan\sqrt{\frac{(1-\xi_2)^{\frac{1}{1-\gamma}}-1}{A(\varphi_m)}}.
\end{aligned}
\tag{42}
$$

7 Importance Sampling using the distribution of visible normals

E. Heitz and E. D'Eon propose to improve the importance sampling of micro-facets based BSDF [9]. Instead of using the normal distribution directly (as presented in Section 6), they suggest to perform a sampling based on a *pdf* built from the distribution of visible normals. Unfortunately, STD cannot be used directly for with this approach. The process requires further investigation to propose an elegant solution, which we leave for future work. We have chosen to describe in this section the locks appearing when trying to blend importance sampling with the distribution of visible normals proposed in [9] and our STD.

7.1 Mathematical background

We use the same notations as in [9] and its associated supplemental material file. The reader may refer to these latter documents for more details.

First, the STD distribution of slopes is defined with $\sigma = 1$ as:

$$P^{22}(p,q) = \frac{1}{\pi}\frac{1}{(1+\frac{p^2}{\gamma-1}+\frac{q^2}{\gamma-1})^\gamma}$$

with p and q the slopes of the corresponding micro-facet normal. p is first sampled following the one-dimensional distribution of visible slopes $P^2_{\omega_i}(p)$. The one-dimensional distribution of slopes $P^2(p)$ is given by:

$$\begin{aligned} P^2(p) &= \int_{-\infty}^{\infty} P^{22}(p,q)\ dq \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{(1+\frac{p^2}{\gamma-1}+\frac{q^2}{\gamma-1})^\gamma}\ dq \\ &= \frac{1}{\sqrt{\pi}}\frac{(\gamma-1)^\gamma\Gamma(\gamma-\frac{1}{2})}{\Gamma(\gamma)(\gamma-1+p^2)^{\gamma-1/2}}, \end{aligned}$$

and $P^2_{\omega_i}(p)$ is

$$\begin{aligned} P^2_{\omega_i}(p) &= \frac{(-p\sin\theta_i+\cos\theta_i)\chi^+(-p\sin\theta_i+\cos\theta_i)P^{2-}(p)}{\int_{-\infty}^{\infty}(-p'\sin\theta_i+\cos\theta_i)\chi^+(-p'\sin\theta_i+\cos\theta_i)P^{2-}(p')\ dp'} \\ &= \frac{G_1(\omega_i)}{\cos\theta_i}(-p\sin\theta_i+\cos\theta_i)\chi^+(-p\sin\theta_i+\cos\theta_i)\frac{(\gamma-1)^\gamma\Gamma(\gamma-\frac{1}{2})}{\sqrt{\pi}\Gamma(\gamma)(\gamma-1+p^2)^{\gamma-1/2}}, \end{aligned}$$

with θ_i the elevation angle corresponding to the incident direction ω_i. The associated *cdf* is:

$$\begin{aligned} C^2_{\omega_i}(p) &= \int_{-\infty}^{p} P^{2-}_{\omega_i}(p')\ dp' \\ &= \frac{G_1(\omega_i)(\gamma-1)^\gamma\Gamma(\gamma-\frac{1}{2})}{\sqrt{\pi}\Gamma(\gamma)}\left((\gamma-1)^{\gamma-1/2}\ p\ {}_2F_1\left(\frac{1}{2},\gamma-\frac{1}{2},\frac{3}{2},\frac{-p^2}{\gamma-1}\right)\right. \\ &+ \left.\frac{\sqrt{\pi}\Gamma(\gamma)}{2(\gamma-1)^\gamma\Gamma(\gamma-1/2)}+\frac{\tan\theta_i}{(2\gamma-3)(p^2+\gamma-1)^{\gamma-3/2}}\right). \end{aligned}$$

Unfortunately, this *cdf* is non invertible in the general case due to the special function ${}_2F_1$. A first idea should be to precompute and tabulate its values for integer and half-integer values of γ but the

GAF G_1 remains challenging to handle.

The second part of the sampling process proposed in [9] is to sample the q slope knowing p by using the conditional *pdf* $P^{2|2}(q|p)$

$$
\begin{aligned}
P^{2|2}(q|p) &= \frac{P^{22}(p,q)}{\int_{-\infty}^{\infty} P^{22}(p,q')\ dq'} \\
&= \frac{\frac{1}{(1+\frac{p^2}{\gamma-1}+\frac{q^2}{\gamma-1})^\gamma}}{\int_{-\infty}^{\infty} \frac{1}{(1+\frac{p^2}{\gamma-1}+\frac{q'^2}{\gamma-1})^\gamma}\ dq'}.
\end{aligned}
$$

The associated conditional *cdf* is:

$$
\begin{aligned}
C^{2|2}(q|p) &= \frac{\int_{-\infty}^{q} \frac{1}{(1+\frac{p^2}{\gamma-1}+\frac{q'^2}{\gamma-1})^\gamma}\ dq'}{\int_{-\infty}^{\infty} \frac{1}{(1+\frac{p^2}{\gamma-1}+\frac{q'^2}{\gamma-1})^\gamma}\ dq'} \\
&= \frac{\int_{-\infty}^{q} \frac{1}{(\frac{1}{\gamma-1}+p^2+q'^2)^\gamma}\ dq'}{\int_{-\infty}^{\infty} \frac{1}{(\frac{1}{\gamma-1}+p^2+q'^2)^\gamma}\ dq'}.
\end{aligned}
$$

Based on the change of variable $a = \frac{1}{\gamma-1} + p^2$, the formulation becomes independent of p:

$$
C^{2|2}(q|p) = \frac{\int_{-\infty}^{q} \frac{1}{(1+\frac{q'^2}{a})^\gamma}\ dq'}{\int_{-\infty}^{\infty} \frac{1}{(1+\frac{q'^2}{a})^\gamma}\ dq'}.
$$

Let be $z = \frac{q}{\sqrt{a}}$ and $z' = \frac{q'}{\sqrt{a}}$, we obtain:

$$
\begin{aligned}
C^{2|2}(q|p) = C_z(z,\gamma) &= \frac{\int_{-\infty}^{z} \frac{1}{(1+z'^2)^\gamma}\ dq'}{\int_{-\infty}^{\infty} \frac{1}{(1+z'^2)^\gamma}\ dq'}. \\
&= \frac{1}{2} + \frac{z\Gamma(\gamma)\ {}_2F_1\left(\frac{1}{2},\gamma,\frac{3}{2},-z^2\right)}{\sqrt{\pi}\Gamma(\gamma-\frac{1}{2})}.
\end{aligned}
$$

Here again, the *cdf* is not directly invertible, which is also the case with GGX. Heitz and D'Eon [9] propose a rational polynomial to fit C_z^{-1} which is conceivable for integer and half-integer values of $\gamma \in [2, 50]$. Note that to do this, the first blocking point about the p sampling, previously presented, must be solved.

7.2 Pseudo importance sampling using the visible normals distribution

Importance sampling using the visible normals distribution (called VNIS in this section) drastically improves rendering and must be implemented to make fully practicable the proposed distribution. As mentioned previously, find a closed form for VNIS with STD remains challenging. We performed some experiments about the VNIS efficiency with GGX and Beckmann (Figure 4) and made the following observations. Firstly, the variance reduction is improved when the roughness σ increases. When $\sigma \leq 0.01$, the variance reduction is low and normal importance sampling compared to VNIS offers quasi-identical quality results. Secondly, GGX and Beckmann are special cases of STD with

$\gamma = 2$ and $\gamma > 40$ respectively and VNIS from these distributions could be used for larger set of values of γ. Using Beckmann VNIS offers pleasant results for $\gamma > 10$ independently of the values used for σ ($\sigma > 0.01$) used (even if the quality increases according to roughness). GGX VNIS could be used for $1.5 < \gamma \leq 5$ with $\sigma > 0.3$. Two gaps appear: one for $0.01 < \sigma \leq 0.3$ and another for $5 < \gamma \leq 10$. In that cases, VNIS strategy must be carefully chosen otherwise the quality results is worse than with normal importance sampling. During our experiments, we observed that Beckmann VNIS can be used from $\gamma \geq 5$ with $\sigma = 0.3$ without lowering the quality compared to normal importance sampling. This lower bound of the Beckmann VNIS can be controlled by a simple linear interpolation. Concerning GGX VNIS, good sampling is obtained for $\gamma = 5$ and $\sigma = 0.3$. We adopt the same strategy as for Beckmann VNIS to control the upper bound of the use of GGX VNIS. Starting from these observations, we propose the simple pseudo-VNIS algorithm 1 which makes the STD distribution practical. When good sampling is not guaranteed by GGX or Beckmann VNIS, a simple importance sampling is performed as described in Section 6. Moreover, the weight with uncorrelated GAF cannot be simplified as for GGX and Beckman and it stays:

$$weight(v) = \frac{G}{G_1(v)} \tag{43}$$

Some results obtained with this simple method are shown in Figure 5. Note that this experiment must be deepened but we think it opens new insights for future work on STD VNIS.

Algorithm 1 Pseudo-VNIS algorithm

if $\sigma \leq 0.01$ **then**
    Perform normal sampling as explained in Section 6
else if $\gamma > 10$ **then**
    Perform VNIS with Beckmann distribution
else
    GGX upper bound $\leftarrow$ $min\left(5,\ 2 + \frac{\sigma}{0.1}\right)$
    Beckmann lower bound $\leftarrow$ $max\left(5,\ 10 - 5 \times \frac{max(0,\ \sigma - 0.2)}{0.1}\right)$
    if $\gamma <$ GGX upper bound **then**
        Perform VNIS with GGX distribution
    else if $\gamma >$ Beckmann lower bound **then**
        Perform VNIS with Beckmann distribution
    else
        Perform normal sampling as explained in Section 6
    end if
end if

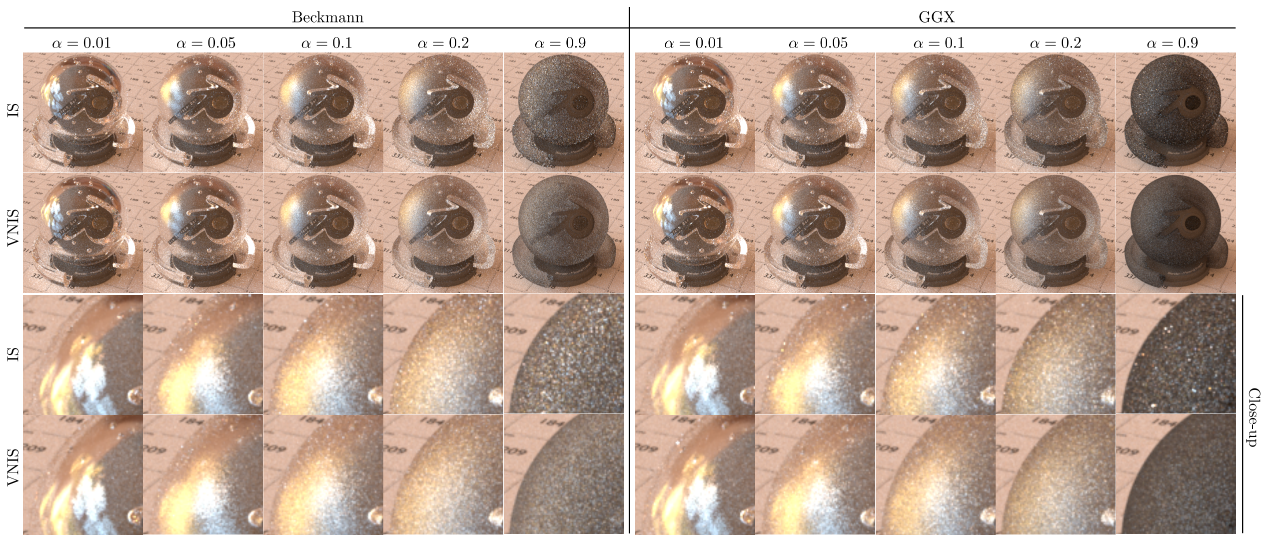

Figure 4: VNIS (64 samples per pixel) for rough dielectric material with the Beckmann and GGX distributions (please refer to the digital version for accurate visualization and zoom on picture to distinguish the MC noise).

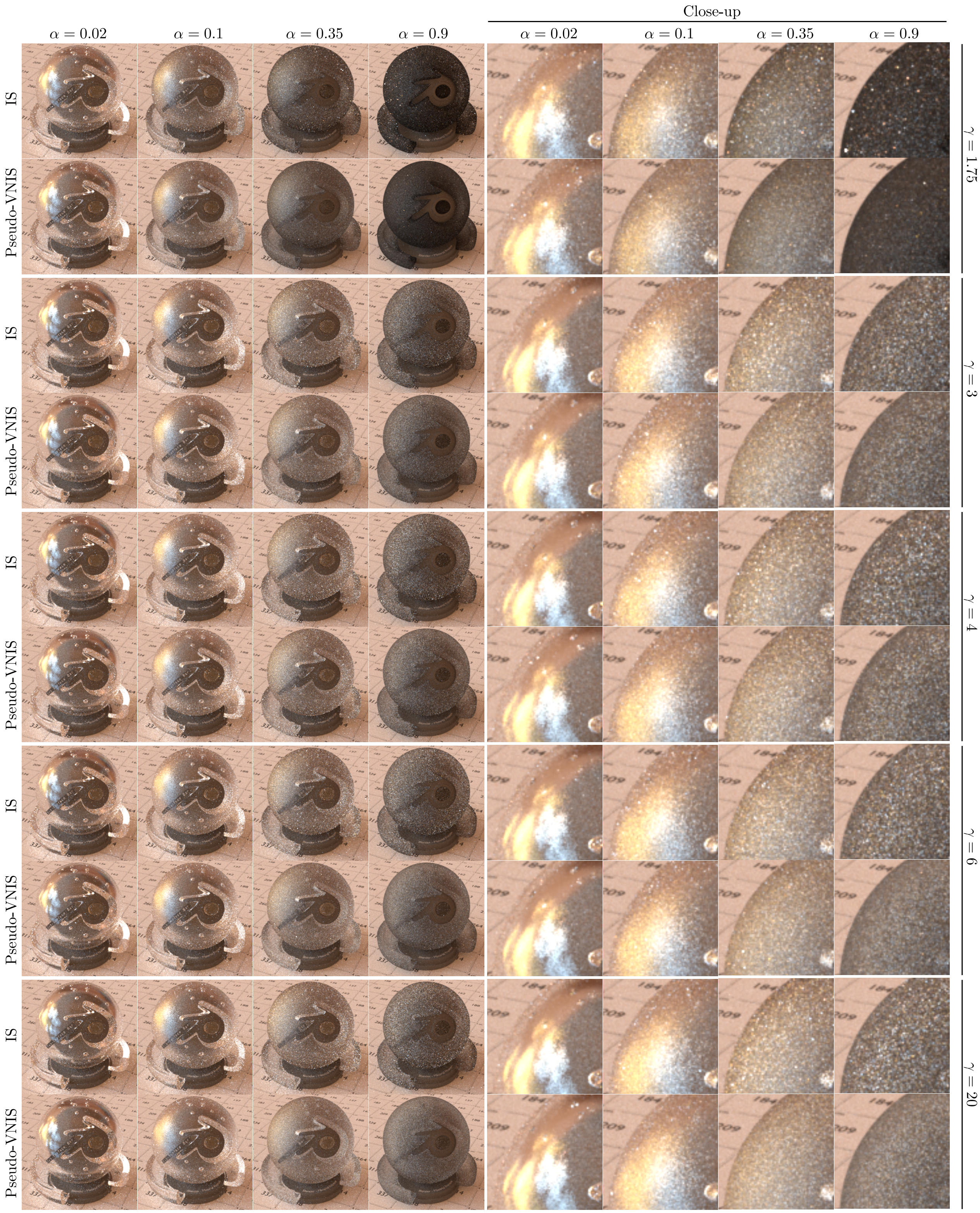

Figure 5: Pseudo-VNIS (64 samples per pixel) with algorithm 1 with rough dielectric material for different values of γ (please refer to the digital version for accurate visualization and zoom on picture to distinguish the MC noise).

8 Implementation details for the ${}_2F_1$ function

The special function ${}_2F_1\left(\frac{1}{2}, \gamma - \frac{1}{2}, \frac{3}{2}, z\right)$ appearing in the Equation 21 are handled by the *GNU Scientific Library* [7] (GSL), and the Gauss hypergeometric function is defined for $z \in]-1, 1]$ in GSL. Unfortunately, in our case, $z = \frac{-\cot^2\theta}{(\gamma-1)\sigma^2}$ and takes its value on $]-\infty, 0]$. In such a case, linear transformations can be derived to rescale the fourth parameter of the function ${}_2F_1$ in $]-1, 1]$ [1].

Using some of these transformations, we propose an algorithm to compute the ${}_2F_1$ function. The thresholds used in the if-statements in lines 6 and 10 prevent from the non convergence of computing due to the maximum iteration number fixed in GSL.

Algorithm 2 Calculate the Gauss hypergeometric function ${}_2F_1\left(\frac{1}{2}, \gamma - \frac{1}{2}, \frac{3}{2}, z\right)$ with GSL.

Require: $z \in]-\infty, 0]$

1: **if** $z == 0$ **then**
2:    **return** ∞
3: **end if**
4: **if** $z \leq -1$ **then**
5:    $normalZ \leftarrow z/(z-1)$ // Normalize the z parameter
6:    **if** $\gamma \geq 12 \mid\mid normalZ \geq 0.99$ **then** // use formula 15.3.7 in [1]
7:       $x \leftarrow \frac{\Gamma(\frac{3}{2})\Gamma(\gamma)}{\Gamma(\gamma-\frac{1}{2})}(-z)^{-a}$ gsl_sf_hyperg_2F1$(\frac{1}{2}, \gamma - \frac{1}{2}, 2 - \gamma, \frac{1}{z})$
8:       $y \leftarrow \frac{\Gamma(\frac{3}{2})\Gamma(1-\gamma)}{\Gamma(\frac{1}{2})\Gamma(2-\gamma)}(-z)^{-b}$gsl_sf_hyperg_2F1$(\gamma - \frac{1}{2}, \gamma - 1, \gamma, \frac{1}{z})$
9:       **return** $x + y$
10:    **else if** $\gamma = 2 \mid\mid \gamma \geq 12$ **then** // use formula 15.3.5 in [1]
11:       **return** $(1-z)^{-b}$ gsl_sf_hyperg_2F1$(\gamma - \frac{1}{2}, 1, \frac{3}{2}, normalZ)$
12:    **else** // use formula 15.3.4 in [1]
13:       **return** $(1-z)^{-a}$ gsl_sf_hyperg_2F1$(\gamma - \frac{1}{2}, 2 - \gamma, \frac{3}{2}, normalZ)$
14:    **end if**
15: **else** // Use directly the function provided in GSL
16:    **return** gsl_sf_hyperg_2F1$(\frac{1}{2}, \gamma - \frac{1}{2}, \frac{3}{2}, z)$
17: **end if**

References

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