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A concise b-rep data structure for stratified subanalytic objects

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Abstract

Current geometric kernels suffer from poor abstraction and design of their data structures. In part, this is due to the lack of a general mathematical framework for geometric modelling and processing. As a result, there is a proliferation of ad hoc solutions, say external data structures, whenever new problems arise in industry, causing serious difficulties in software maintenance. This paper proposes such a framework based on subanalytic geometry and theory of stratifications, as well as a concise data structure for it, called DiX (Data in Xtratus). Basically, this is a non-manifold b-rep (boundary representation) data structure without oriented cells (e.g. half-edges, co-edges or so). Thus, it is more concise than the traditional b-rep data structures provided that no oriented cells (e.g. half-edges, half-faces, etc.) are used at all. Nevertheless, all the adjacency and incidence data we need is retrieved by a single operator, called mask operator. Besides, the DiX data structure includes shape descriptors, as generalizations of loops and shells, to support shape reasoning as needed in the design and implementation of shape operators such as, for example, Euler operators.

1. Introduction

There are two major ways to describe a geometric object. It can be viewed as either a point set X or a point set X partitioned into a set $\{X_i\}$ of point subsets in some Euclidean space. The first view originated the development of the class of geometric models known as CSG (Constructive Solid Geometry) models in the 70's. The second gave rise to the class of b-rep (boundary representation) models about the same time.

2. Theory of subanalytic objects

2.1. Semialgebraic sets

After polyhedra, the simplest geometry used in geometric kernels is the semialgebraic geometry. By abuse of language, we say that the semialgebraic geometry consists of semialgebraic point sets. A semialgebraic set $X \subseteq \mathbb{R}^m$ is defined by the intersecting set of a family $\{x \in \mathbb{R}^m : f(x) \geq 0\}$ of sets described by polynomial equalities (zero sets) and inequalities (positive or negative sets), being then $f : \mathbb{R}^m \to \mathbb{R}$ a polynomial function. Semialgebraic were incorporated into the pioneering CSG geometric kernels to construct semialgebraic solid primitives (e.g. sphere, cylinder, cone, block, wedge), as well as more complex semialgebraic solids embodying mechanical components and parts, through set-theoretic operations. Semialgebraic subsets of $\mathbb{R}^m$ are closed under finite unions ($\cup$), finite intersections ($\cap$) and complements ($\setminus$) (and difference of any two), i.e. they form a closed Boolean algebra[6]. That is, applying any of these operators to two or more semialgebraic sets has always as a result another semialgebraic set. Assuming that the the interior $\mathrm{Int}(X)$ is well-defined, it follows that the boundary $\mathrm{Bd}(X) = X \setminus \mathrm{Int}(X)$, frontier $\mathrm{Fr}(X) = \mathrm{Bd}(X) \cup \mathrm{Bd}(\mathbb{R}^m \setminus X)$, and closure $\mathrm{Cl}(X) = \mathrm{Int}(X) \cup \mathrm{Fr}(X)$ of a semialgebraic set X in $\mathbb{R}^m$ is also a semialgebraic set. For example, the open disc $X = \mathbb{D}^2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ in $\mathbb{R}^2$ coincides with its interior; hence, $\mathrm{Bd}(\mathbb{D}^2) = \varnothing$, $\mathrm{Fr}(\mathbb{D}^2) = \mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, and $\mathrm{Cl}(\mathbb{D}^2) = \mathrm{Fr}(\mathbb{D}^2)$. Note that, in general, $\mathrm{Bd}(X) = \mathrm{Fr}(X)$ only for closed sets.

According to the definition above, a semialgebraic set is an *implicit* semialgebraic point set because it is formed by intersecting zero sets and positive (or negative) sets of a family of functions or mappings. Remarkably, this does not exclude *parametric* point sets defined by polynomials such as, for example, Bézier curves and surfaces predominant in the automotive and aerospace industries. These parametric curves and surfaces are *images* of some semialgebraic set.

But, a theorem due to Tarski and Seidenberg[22, 17] states that the *image* of a semialgebraic subset of $\mathbb{R}^m$ under a polynomial mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ is semialgebraic. This formalises our intuition that either implicit or parametric point sets defined polynomials are semialgebraic. Tarski and Seidenberg also proved that semialgebraicity is preserved by rational maps, i.e. if $X \subseteq \mathbb{R}^m$ is semialgebraic, and $f : X \to \mathbb{R}^n$ is rational, then $f(X)$ is also semialgebraic in $\mathbb{R}^n$. This means that NURBS (Non-Uniform Rational B-Splines) are also semialgebraic curves and surfaces.

2.2. Semianalytic sets

Unfortunately, the semialgebraic sets do not include useful functions such as, for example, the trigonometric functions necessary to define screw threads and springs. The natural generalisation comprises the real analytic functions. They include polynomial, rational, and transcendental functions. Their analyticity guarantees that they can be uniquely approximated by the Taylor series expansion. This makes possible important geometric operations such as, for example, stitching surface patches with C^r-contact smoothness. Real analytic functions define semianalytic sets, as polynomial functions define semialgebraic sets, i.e. through equalities and inequalities of real analytic functions. With the exception of the direct image property, semianalytic sets enjoy all the good properties of semialgebraic sets. In fact, semianalyticity is preserved by Boolean and topological operations, yet it is not preserved by projection operations (see Lojasiewicz[12] for an example). Tarski-Seidenberg theorem then fails, i.e. the direct image of a semianalytic set under a real analytic mapping may be a semianalytic set or not. Thus, there is no guarantee that a parametric curve or surface as direct image of a semianalytic line segment or surface patch under a real analytic mapping is semianalytic. Nevertheless, the image of any semianalytic set by an analytic isomorphism (i.e. a diffeomorphism) is semianalytic [11].

2.3. Subanalytic sets

Semianalytic and subanalytic sets coincide up to dimension 2, and start to differ from each other from the dimension 3 up. See Lojasiewicz[12] for a subanalytic set in $\mathbb{R}^3$ that is not semianalytic. Following the definition given by Hironaka[5], subanalytic sets include both semianalytic sets and their images by proper real analytic sets. That is, the Tarski-Seidenberg theorem is valid. Subanalytic sets then provide us with a self-contained, general geometric framework encompassing both implicit and parametric point sets defined by real analytic functions or mappings.

3. Regular stratified objects

The importance of the subanalytic sets also comes from the fact they can be stratified somehow. Roughly speaking, a stratification is a partition of a point set into point subsets, being each a subset of a manifold of some dimension. (By definition, an *n-dimensional manifold* is a topological space homeomorphic to $\mathbb{R}^n$ *locally*, i.e. it looks $\mathbb{R}^n$ in the neighbourhood of any of its points.) These manifold constituents of a stratification are called strata. For example, a cube can be subdivided or stratified into six faces (or 2-dimensional strata), twelve edges (or 1-dimensional strata), and eight vertices (or 0-dimensional strata), as well as its solid interior (or 3-dimensional stratum).

Without stratifications it is not possible to fulfil important application requirements such as, for example, graphical interaction with objects by means of picking-up operations on strata, stratum-by-stratum construction of n-dimensional objects, meshing for computer games and finite-element modelling, just to mention a few.

3.1. Notion of stratification

There are many ways of partitioning a point set into subsidiary point subsets that are manifolds. But, not all partitions are stratifications. A stratification of a set $X \subset \mathbb{R}^n$ is a partition of X into smooth (i.e. C^∞) sub-manifolds X_i of $\mathbb{R}^n$, called strata, such that the family $\{X_i\}$ is locally finite at each point of X [19]. For a positive integer r, we also define a C^r stratification. By "locally finite" we mean that each point has a neighbourhood meeting only finitely many strata. Thus, this definition of stratification includes three conditions, namely:

1. *Manifoldness condition*. Strata are all submanifolds. By definition, a submanifold has no boundary, but as a stratum it may be bounded by lower dimensional strata in X. For example, in a stratified cube, its interior 3-dimensional stratum is bounded by n-dimensional strata ($0 \leq n \leq 2$), i.e. vertices, edges, and faces, respectively.
2. *Smoothness condition*. Every sub-manifold is smooth, no matter whether it is an implicit submanifold or a parametric submanifold. Unlike smooth implicit submanifolds, smooth parametric submanifolds admit self-intersections. For example, a Bézier curve with self-intersections is smooth since such self-intersections occur in distinct parameterisation sub-intervals. In Figure 1(a), the self-intersecting face f_1 is not smooth if it is implicitly-defined by the set-theoretic difference between $x^2 - zy^2 = 0$ and $z < 0$. This means that the partition in Figure 1(a) is not a stratification. But, if the face f_1 is parametrically-defined by $f(u,v) = (x(u,v), y(u,v), z(u,v)) = (uv, u, v2)$, then it is smooth, and the partition in Figure 1(a) can be considered as a stratification. In fact, the rank of the Jacobian

$$\mathrm{J}f(u,v) = \begin{bmatrix} v & u \\ 1 & 0 \\ 0 & 2v \end{bmatrix}$$

only drops below 2 at $(0,0)$. That is, the point (0,0) is the only point at which the tangent plane is not defined.

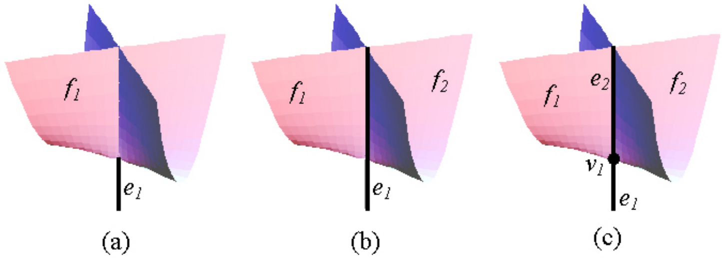

Figure 1: *Three partitions of the Cartan umbrella* $x^2 - zy^2 = 0$.

Any other point on the positive z-axis has a parameterised neighbourhood that can be approximated by a tangent plane in relation to the parameterisation. But, we know that all points $(0,0,v^2)$ along the positive z-axis are image double points after folding the v-axis of $\mathbb{R}^2$ at the origin $(0,0)$. That is, the positive z-axis is a self-intersecting point set that is not detected by the Jacobian. In a word, smooth implicit strata have no singularities, but smooth parametric strata may possess some. A smooth stratum without singularities is said to be *regular*. In other words, smoothness and regularity coincide for implicit, but not for parametric, submanifolds.

3. *Local finiteness condition.* A stratification with a finite number of strata is called finite. For example, the stratifications in Figure 1(b) and (c) are finite. In particular, the stratified object in Figure 1(b) has two strata $X_1 = \{e_1\}$ and $X_2 = \{f_1, f_2\}$, with the component e_1 of X_1 separating the components f_1, f_2 of X_2. Even considering that all strata are connected, we obtain a stratification with three strata, namely: $X_1 = \{e_1\}$, $X_2 = \{f_1\}$, and $X_2 = \{f_2\}$. A stratification of a set X is said to be *locally finite* if, for any point $x \in X$, there exists a neighbourhood intersecting a finite number of strata. Finiteness implies local finiteness, but not the other way round.

In Figure 1, we have three partitions of the Cartan umbrella defined by the zero-set $x^2 - zy^2 = 0$. The partition (a) has one edge e_1 and one self-intersecting face f_1. It is not a stratification because f_1 is not a 2-dimensional (implicit) sub-manifold. But, the partitions (b) and (c) are both stratifications. The stratification (b) has one edge e_1 and two faces f_1 and f_2, but no vertices, while the stratification (c) consists of one vertex v_1, two edges e_1 and e_2, and two faces f_1 and f_2. In fact, their constituent subsets are all sub-manifolds (*manifoldness condition*) of a finite family, what implies that the *local finiteness condition* is also satisfied.

However, local finiteness property does not imply that a family of submanifolds is finite. In a local finite family of submanifolds, each submanifold meets (i.e. is part of) only finitely many closures of sub-manifolds in the same partition. For example, let us consider a cylindrical surface with both top and bottom circular edges partitioned into an infinite number of vertices (points). The cylindrical surface itself is partitioned into an infinite number of vertical edges, each connecting a top vertex to a bottom vertex. This stratification has an infinite number of strata, but even so it is locally finite. In fact, every vertex meets (is in the closure of) two strata: a paralleled edge and a face (i.e. either top or bottom face), whereas no edge or face meets any higher dimensional strata.

3.2. (Weak) frontier condition

Manifoldness, C^r smoothness, and local finiteness are all necessary but not sufficient conditions for geometric modelling purposes. In particular, they do not guarantee the existence of correct traversal algorithms for stratified objects represented in b-rep (boundary representation) data structures. For that, it is necessary that each stratum has a well-defined frontier and its topological type be invariant locally. By a well-defined frontier of an n-stratum we mean a set of strata of dimension less than n in its closure. Stratifications having all strata with well-defined frontiers are said to satisfy the "frontier condition". The *frontier condition* states that if one stratum is partly in the closure of another stratum, then it is entirely in the closure of that other stratum[23].

Let us look again at Figure 1. Let us assume that all strata are connected. The stratification (b) fails to satisfy the frontier condition viewing that $X_1 = \{e_1\}$ is partly, but not entirely, in the closure of $X_2 = \{f_1, f_2\}$. On the contrary, the stratification (c) satisfies the frontier condition. But, if e_1 and e_2 in the stratification (c) were considered as two components of a common 1-stratum, the frontier condition is violated because e_2 would not be in the frontier of f_1 and f_2. Instead, the stratification (c) would satisfy the *weak* frontier condition.

A stratification satisfies the *weak frontier condition* if the family of the connected components of all X_i satisfies the frontier condition[19]. That is, the weak frontier condition is for stratum components as the frontier condition is for their strata. The connectedness property is closely attached to application requirements. Some applications need multi-connected strata[20]. For example, in form feature-based mod-

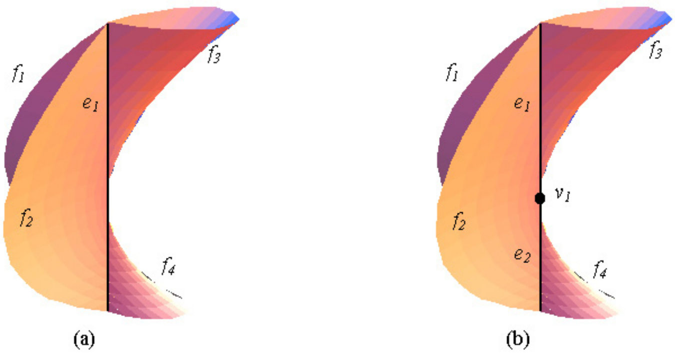

Figure 2: *Two stratifications of the algebraic variety* $y^2 - zx^2 + x^3 = 0$.

elling, having multi-component strata with the same supporting geometry would facilitate the development of shape recognition and reconstruction algorithms, as well as undo operations. But, most applications use only connected strata. This explains why b-rep data structures use only connected strata. Therefore, a geometric kernel must reinforce the weak frontier condition, instead of the frontier condition. Besides, the weak frontier condition has advantage over the frontier condition because it guarantees that a stratification is (topologically) equisingular or regular, i.e. all points of each stratum component have the same topological type.

Figure 2 shows two stratifications of the algebraic variety $X = \{(x,y,z) \in \mathbb{R}^3 : y^2 - zx^2 + x^3 = 0\}$. In Figure 2(a), let $X_1 = \{e_1\}$ denote the z-axis and let $X_2 = \{f_1, f_2, f_3, f_4\}$ be the complement of X_1 in X. Then $\{X_1, X_2\}$ is a stratification of X. For each $p \in X_1$, let N_p be the normal plane to X_1 at p. Then $\{X_1, X_2\}$ would be (topologically) equisingular if the germs $(N_p \cap X_2, p)$ were topologically equivalent. They are not, but are for $p \neq (0,0,0)$. The stratification $\{X_1, X_2\}$ satisfies the frontier condition, but not the weak frontier condition. The weak frontier condition is violated because, for example, $e_1 \cap \mathrm{Cl}(f_3) \neq \varnothing$, but $e_1 \not\subset \mathrm{Cl}(f_3)$. On contrary, the stratification $\{X_1, X_2, X_3\}$ in Figure 2(b) is equisingular, with $X_1 = \{v_1\}$, $X_2 = \{e_1, e_2\}$, and $X_3 = \{f_1, f_2, f_3, f_4\}$. It satisfies both the frontier condition and the weak frontier condition.

This suggests that topologically regular stratifications satisfy the weak frontier condition. This is important because it meets our application requirements that each stratum of a stratified set should consist of equally bad points. Otherwise, incidence-based traversal algorithms as those embedded into Euler operators and combinatorial b-rep data structures are difficult to be conceived.

3.3. Types of stratification

Many sorts of stratifications are admissible for subanalytic sets[15], but not all fulfil the requirements of most geometric applications. In fact, as suggested above, not all types of stratification are adequate as mathematical models for the design and implementation of B-Rep data structures.

A stratification distinguishes from another by imposing different regularity conditions on the strata. A regularity condition determines how the strata fit together, i.e. their adjacency or incidence relationships. In abstract terms, for each pair X_i, X_j of strata and each $p \in X_i \cap \mathrm{Fr}(X_j)$, we have to guarantee that X_j is Ω-regular over X_i at p. The term 'Ω-regular' may have several particular meanings, each for a distinct stratification. For example, a 'topologically-regular' stratification of X is one for which the topological type remains invariant for all points of each stratum X_i or stratum component c_i with respect to X, and thus with relation to any incident stratum X_j or stratum component c_j, respectively. A stratification of X that has stratum components in which each point has the same topological type with respect to X is called weak topological stratification, i.e. it satisfies the weak frontier condition[4]. A strong topological stratification of X is defined in terms of strata rather than stratum components. Thus, all points of each one of its strata have the same topological type with respect to X, i.e a strong topological stratification obeys the frontier condition[15].

Unfortunately, stratifications driven by topological criteria are useless from a computational point of view. In fact, in practice, they provide no algebraic machinery (i.e. a computational algorithm) to resolve or detect singularities, and then partition a point set X accordingly. Even a C^r-regular stratification (i.e. a differential stratification) may not satisfy the weak frontier condition.

Let us consider again the Cartan umbrella $X = \{(x,y,z) \in \mathbb{R}^3 : x^2 - zy^2 = 0\}$. It is the zero-set of $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x,y,z) = x^2 - zy^2$. We know that the maximal rank of its Jacobian $\mathrm{J}f = [\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\ \frac{\partial f}{\partial z}] = [2x\ \ -2zy\ \ -y^2]$ is 1. So, the co-rank is 0 for the regular subset of X (i.e. its two umbrella sheets) or 1 for its singular subset (i.e. the z-axis). Thus, the co-rank cannot increase any further, and, consequently, the origin (0,0,0) is not distinguishable from other singular points in the singular subset of X. The corresponding C^1-regular stratification of X is given in Figure 1(b).

It is then desirable to have some kind of *geometric* regularity criterion for stratifications that would also satisfy the weak frontier condition. The well-known *Whitney condition* is one such geometric regularity criterion. It states that stratum X_j is Whitney-regular over stratum X_i at $p \in X_i \cap \mathrm{Fr}(X_j)$, if and only if for any point $q \in X_j$ near p, the tangent space T_qX_j (of q in X_j) nearly contains T_pX_i and the vector $\overrightarrow{pq}$. Therefore, a *Whitney-regular stratification* is a stratification that satisfies the Whitney regularity condition, i.e. for each pair of strata X_i and X_j, and for all $p \in X_i \cap \mathrm{Fr}(X_j)$, X_j is Whitney regular over X_i at p . (See Shiota[19], Wall[25], Shapiro[18], Middleditch[15], and Gomes[4] for more details about Whitney stratifications.) Remarkably, as proved in Mather[14], Whitney-regularity plus local finiteness implies the frontier condition.

However, Whitney stratifications do not guarantee that along a stratum X_i the local topological type remains invariant in X, unless all X_i are connected. Fortunately, there is a sort of geometric stratification, called *Verdier stratification*, which is topologically equisingular along each connected component of each stratum[23]. Its regularity condition states that every pair (X_i, X_j) of strata is Verdier-regular at $p \in X_j$ if there exists $\varepsilon > 0$ and a neighbourhood N_p of p such that $\mathrm{dist}(\mathrm{T}_yX_j, \mathrm{T}_xX_i) \leq \varepsilon||y-x||$, for all $y \in X_j \cap N_p$ and $x \in X_i \cap N_p$. (Here $\mathrm{dist}(A,B)$ measures "the distance between vector subspaces".) For example, the object in Figure 2(b) is Verdier-stratified. As proved in Loi[10], the Verdier condition implies the Whitney condition, and thus Verdier stratifications refine Whitney stratifications.

We have just came to a point at which the relationship between geometry and topology, as usual in b-rep data structures, can be formally validated by the following theorem due to Verdier[24, 23]:

Theorem 1 Every subanalytic set is Verdier-stratifiable. ■

Therefore, a *subanalytic* Verdier stratification is a stratification in which all strata are subanalytic submanifolds of $\mathbb{R}^n$. It formalizes the relation between the *geometry* (e.g. the subanalytic geometry) and the *topology* (e.g. stratum complex or stratification) found in the b-rep data structures.

3.4. Stratified sets versus cell complexes

In a way, stratified sets generalize cell complexes from mathematics in that an n-cell is a particular n-stratum that is homeomorphic to $\mathbb{R}^n$, i.e. an n-stratum without holes (see next section for more details). Unlike a cell, a stratum need not be connected, nor bounded, nor globally homeomorphic to an open ball. For example, the 1-sphere $\mathbb{S}^1 = \{p \in \mathbb{R}^2 : ||p|| = 1\}$ admits a stratification of a single 1-stratum, while a cell complex requires at least two cells, e.g. a 1-cell and a 0-cell. Although most B-Rep data structures represent stratified sets, not just cell complexes, they lack generality even in $\mathbb{R}^3$. Usually, the construction of a solid object starts with a topological surface homeomorphic to $\mathbb{S}^2 = \{p \in \mathbb{R}^3 : ||p|| = 1\}$ stratified into a 2-cell and a 0-cell by calling the Euler operator `mvfS` (make vertex, face, and shell). (See, for example, Mäntylä[13] for the design and implementation of a minimal set of Euler operators.) This is so because the mathematical model ruled by the Euler-Poincaré formula does not include strata homeomorphic to spheres; hence, the inclusion of a vertex. On the contrary, the mathematical model proposed in this paper admits strata homeomorphic to spheres or even tori, being it ruled by a more general Euler-Poincaré formula.

4. Shape framework

Understanding shape of point sets, either they are stratified or not, is crucial in the design of b-rep geometric data structures and their shape operators (e.g. Euler operators). There are many types of shape. In the last two sections, we have dealt with four types of shape somehow, namely:

1. *Geometric shape*. It is the rigid shape of point sets. It is defined by isometries (Euclidean geometry), polynomials (algebraic geometry), rationals (rational geometry), transcendentals (transcendental geometry), and more generally analytic functions (semianalytic and subanalytic geometries). Distance between points is its essential invariant. This enables us to say that a cube and a sphere are geometrically distinct.
2. *Differential shape*. It has to do with the *smooth* shape of point sets. Equi-smooth point sets are defined by diffeomorphisms. For example, the 1-manifolds $X = \{(x,y) \in \mathbb{R}^2 : y = 0\}$ and $Y = \{(x,y) \in \mathbb{R}^2 : y = |x|\}$ are not equally smooth. X is smooth, but Y is only piecewise-smooth because it has a singularity (a corner) at $x = 0$. This suggests that the number of equi-dimensional singularities may work as an invariant for equi-dimensional objects. For example, a cube surface and a paralleliped surface are smoothly equivalent because they possess the same number of 0-singularities (8 vertices) and 1-singularities (12 edges). In fact, diffeomorphic manifolds have diffeomorphic boundaries [7].
3. *Topological shape*. It has to do with the *dimension-constraint deformable* shape of point sets. It is defined by homeomorphims. Two objects are said to be topologically-equivalent or homeomorphic if and only if the first can be elastically deformed into the second, although preserving their dimensions. For example, a spherical surface and a cube surface have the same topological shape or type since there exists a homeomorphism to deform one into another. Dimension is a topological invariant. Thus, vertices, edges, faces, and higher-dimensional strata are not topologically-equivalent shapes. But, not always the dimension is sufficient to topologically distinguish two objects. For example, a straight-line and a curve

with a self-intersection point have the same dimension, but they are topologically distinct because a straight-line has no self-intersection point or cut-point. The number of cut-points is another topological invariant.

4. *Homotopic shape*. It is also known as the global topological shape. This has to do with the *dimension-constraintless deformable* shape of point sets. It is defined by homotopies. Homotopy mappings generalise homeomorphisms by dropping down the point non-coalescence condition. For example, a straight-line can be collapsed into a single point, i.e. they are homotopically-equivalent, i.e. they have the same homotopy type. Consequently, dimension is not a homotopy invariant. Nevertheless, a 1-circle and a point do not possess the same homotopy type because a 1-circle cannot be coalesced into a point without eliminating the hole through the circle. Such a hole can be eliminated by filling in the 1-circle with a 2-disc, or, alternatively, by cutting the 1-circle in such a way that it is transformed into a curve segment homeomorphic to $\mathbb{R}^1$. Thus, a hole always denotes missing material in some point set. The number of n-dimensional holes of an object or manifold is known as the n-dimensional Betti β_n. Thus, objects with different homotopy types have distinct homologous Betti numbers. The homotopic shape of a point set has precisely to do with the k-dimensional holes it possesses. The presence of holes in a point set prevents the point coalescence of a point somehow.

Betti numbers are important invariants in the design and implementation of Euler operators. They play an important role in Euler-Poincaré formulæ for cell complexes and stratified objects. The arrangement of strata and their holes in a stratified object is characterised as follows:

Theorem 2 Let $\mathbf{X} = (X, \Sigma)$ be a regular stratified subanalytic set in $\mathbb{R}^3$, where X is its underlying point set and Σ is its Verdier stratification (or set of strata). The Euler characteristic of Σ is

$$\chi(\Sigma) = v - (e - e_h) + (f - f_h + f_c) - (s - s_h + s_c) + \chi_\infty(\Sigma) \quad (1)$$

with $\chi_\infty(\Sigma) = -(e_\infty) + (f_\infty - f_{\infty h}) - (s_\infty - s_{\infty h} + s_{\infty c})$, and where v, e, f, and s stand for the number of boundary-complete vertices, edges, faces, and solids in X, respectively; e_∞, f_∞, and s_∞ denote the number of boundary-incomplete edges, faces, and solids in X; e_h, f_h, and s_h stand for the number of boundary-complete 1-holes through edges, faces, and solids, respectively; $f_{\infty h}$ and $s_{\infty h}$ stand for the number of boundary-incomplete 1-holes through faces and solids, respectively; f_c and s_c indicate the number of boundary-complete 2-holes for faces and solids, respectively, while $s_{\infty c}$ indicates the boundary-incomplete 2-holes in solids, respectively. ■

According to Euler characteristic (1), no vertex has a hole. However, assuming that strata may have two or more components, a set of three isolated vertices may be viewed as having three components of a single vertex v. In this case, v is said to have two 0-holes because there are two missing paths between between their vertex components. The number of 0-holes equals the number of paths needed to make a manifold connected.

Additionally, an edge admits 1-holes (e_h). An edge with a single 1-hole ($e_h = 1$) has the homotopy type of a 1-sphere $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, i.e. it can be continuously deformed to a circle.

At last, a face may present several 0-holes (i.e. separate face components), 1-holes (i.e. through holes) preventing its homotopic deformation to a 1-sphere, or a 2-hole (i.e. a void). In the latter case, a face is homotopy-equivalent to the 2-sphere $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. In the formula (1), f_h and f_c stand for 1-holes and 2-holes of faces, respectively.

Another example of a face with a 2-hole is the toroidal surface $\mathbb{T}^2$. It also has two 1-holes because we can draw two imaginary loops on it, which are not contractible to a point, nor contractible to each other. It is then said that $\mathbb{T}^2$ has the homotopy type of $\mathbb{S}^1 \times \mathbb{S}^1$, i.e. two loops or rings intersecting a single point. This means that $\mathbb{T}^2$ can be formed by sweeping the first ring $\mathbb{S}^1$ along the second touching ring $\mathbb{S}^1$. Filling in $\mathbb{T}^2$ with an open solid torus (3-manifold) one obtains a closed solid torus (closed 3-manifold). This filling operation makes the 2-hole of $\mathbb{T}^2$ to disappear, as well as one of its 1-holes (the sweeping ring $\mathbb{S}^1$). That is, only one 1-hole (the revolution ring of $\mathbb{T}^2$) remains in closed solid torus.

It is assumed that, as a topological space, an n-dimensional manifold (or, simply, an n-manifold) is a point set topologized by the usual topology in $\mathbb{R}^n$. Thus, every n-manifold is open in $\mathbb{R}^n$, not necessarily bounded, with possibly many k-dimensional holes ($0 \leq k < n$). Unbounded, or equivalently boundary-incomplete, strata appear with the subscript ∞. For example, with the exception of the vertex v_1 in Figure 1(c), all strata are unbounded. Besides, as shown in (1), unbounded strata may also have holes.

The Euler characteristic (1) regulates the manifold strata (the elementary geometric, differential, and topological shapes) of a stratified object, as well as their homotopic shapes (k-dimensional holes) via Betti numbers. Thus, formula (1) provides us with a shape understanding at the stratum level.

To describe the shape of a stratified object as a whole, we use the global Euler characteristic as follows:

Theorem 3 Let $\mathbf{X} = (X, \Sigma)$ a regular stratified subanalytic set in $\mathbb{R}^3$. The Euler characteristic of X is

$$\chi(X) = (C - C_h + C_c) + \chi_\infty(X) \quad (2)$$

with $\chi_\infty(X) = -(E_\infty) + (F_\infty - F_{\infty h}) - (S_\infty - S_{\infty h} + S_{\infty c})$, and where C, C_h, and C_c stand for the number of boundary-complete components, 1-holes, and 2-holes of $|X|$, respectively; E_∞, F_∞, and S_∞ denote the number of boundary-incomplete components for edges, faces, and solids, respectively; $F_{\infty h}$ and $S_{\infty h}$ denote the number of boundary-incomplete 1-holes through face components and solid components, respectively, and $S_{\infty c}$ the number of boundary-incomplete 2-holes in solid components. ■

Note that the global or homotopic shape of the whole space $|X|$ underlying X can be described by the number of 0-holes, 1-holes, ..., k-holes, which are denoted by H^0, H^1, ..., H^k, $0 \leq k \leq n$, respectively.

5. Incidence framework

By definition, a stratum X_i is adjacent to another stratum X_j (symbolically, $X_i \prec X_j$) if X_i is contained in the frontier of X_j (or, equivalently, $\mathrm{Fr}(X_j) \cap X_i \neq \varnothing$) and the dimension of X_i is less than the dimension of X_j; equivalently, one says that X_j is incident on X_i (symbolically, $X_j \succ X_i$). For example, a vertex bounding an edge is said to be adjacent to it, but there may be many edges incident at the same vertex. The adjacency (and incidence) relation is transitive, i.e. if $X_i \prec X_j$ and $X_j \prec X_k$ then $X_i \prec X_k$.

A n-dimensional stratification can be then viewed as $(n+1)$-tuple $\Sigma = \{X^0, X^1, X^2, ..., X^n\}$, where X^0 is a set of vertices, X^1 is a set of edges, X^2 is a set of faces, ..., and X^n is a set of n-strata.

A n-stratum X_j^n is viewed as a pair of adjacent (or bounding) $(n-1)$-strata $\{X_1^{n-1}, X_2^{n-1}, ..., X_i^{n-1}\}$ and incident $(n+1)$-strata $\{X_1^{n+1}, X_2^{n+1}, ..., X_k^{n+1}\}$, i.e. $X_j^n = \{\{X_1^{n-1}, X_2^{n-1}, ..., X_i^{n-1}\}, \{X_1^{n+1}, X_2^{n+1}, ..., X_k^{n+1}\}\}$. So, we have $X^{n-1} \prec X^n$ and $X^{n+1} \succ X^n$ relations embedded in the DiX data structure (Figure 3).

For example, there are only two basic adjacency relations for 2-dimensional stratifications, namely $V \prec E$ and $E \prec F$; $E \succ V$ and $F \succ E$ are their inverse or incidence relations. These four basic relations can be compounded to form nine adjacency relations as introduced by Weiler[26]. So, the $V \rightarrow F$ and $E \rightarrow E$ relations introduced by Weiler can be obtained as follows:

$$V \rightarrow F = (E \succ V) \circ (F \succ E)$$
$$E \rightarrow E = (V \prec E) \circ (E \succ V) \quad .$$

According to Ni and Bloor[16], those four (out of nine) basic relations form the best representation in the class C_4^9 in terms of information retrieval performance. That is, it requires a minimal number of direct and indirect accesses to the data structure to retrieve those four basic incidence relations and the remaining five compound incidence relations, respectively. A direct access query involves a single call to the mask operator, while an indirect access requires two or more calls to the incidence operator (to be described in the next section), i.e. a composed query.

Therefore, the incidence framework of the DiX data structure just accommodates the optimal C_4^9 representation for 2-dimensional stratifications (e.g. meshes), the C_6^{16} representation for 3-dimensional stratifications, and, more generally, the $C_{2n}^{(n+1)^2}$ representation for n-dimensional objects . It keeps the essential adjacency and incidence data that allows us to derive supplementary data to traverse a stratification quickly.

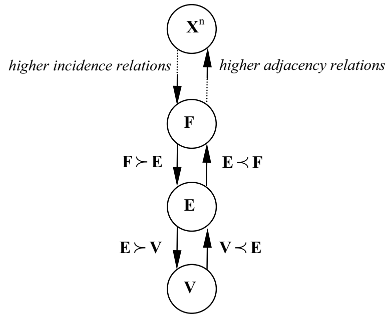

Figure 3: *The incidence framework diagram.*

With these $2n$ adjacency and incidence relations embedded in the DiX data structure, we are able to represent manifold stratifications (Figure 4) and non-manifold stratifications (Figure 5). Besides, the DiX data structure is able to represent unbounded objects with boundary-incomplete strata as those in Figure 1.

6. Incidence operator

Time is a critical factor in many geometric applications. We need fast algorithms to locally query and retrieve strata adjacent to or incident at/on other strata in order to assist design and modelling operations.

Let us then pay attention to the DiX incidence scheme. The incidence scheme of a stratification can be described in terms of a set of stratum-tuples T=$\{(X_i^0, X_j^1, ..., X_l^n)\}$, where there is a transitive adjacency relationship between any two consecutive strata. So, X_i^0 is adjacent to $X_j^1, ..., X_l^n$; alternatively, we say that X_l^n is incident to $X_k^{n-1}, ..., X_j^1, X_i^0$.

For example, the incidence scheme for the stratified object in Figure 5 is as follows:

$$(v_1, e_1, f_1)$$
$$(v_1, e_1, f_2)$$
$$(v_1, e_5, -)$$
$$(v_2, e_2, -)$$
$$(v_2, e_3, -)$$
$$(v_2, e_4, -)$$
$$(v_2, e_5, -)$$
$$(v_3, e_3, -)$$
$$(v_3, -, f_3)$$
$$(v_4, e_2, -)$$
$$(v_5, e_4, -)$$

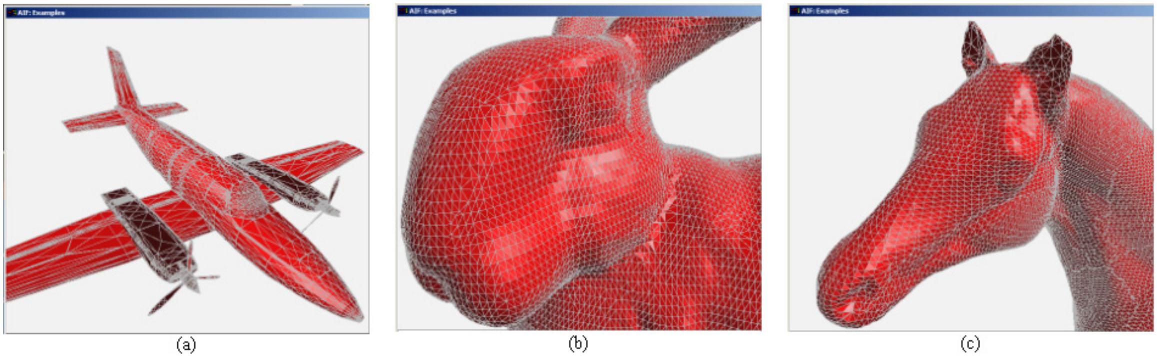

Figure 4: *Some 2-dimensional stratifications.*

This incidence scheme can be considered as a data structure on its own. It was inspired in the cell-tuple data structure due to Brisson[2]. The DiX data structure has the same adjacency and incidence descriptive power as the cell-tuple data structure, but it is more concise and less time-consuming. In fact, the DiX consists of a set of cells (not a set of cell-tuples), each having represented incident and adjacent strata explicitly, what extensively reduces the accessing time to the data structure. Besides, unlike the cell-tuple data structure, the DiX data structure may accommodate non-manifold objects and strata may possess holes.

The DiX data structure uses a single adjacency and incidence operator, called mask operator. The mask operator is defined by $\blacktriangleright\!\!\triangleleft_d : X^0 \times X^1 \times \ldots \times X^n \rightarrow \Sigma$, where Σ is the set of all strata belonging to a stratification, such that $\blacktriangleright\!\!\triangleleft_d(X_i^0, X_j^1, ..., X_l^n) = \{X_k^d\}$, i.e. a set of d-dimensional strata. The arguments of $\blacktriangleright\!\!\triangleleft_d$ are strata in the set $X^0 \times X^1 \times \ldots \times X^n$. A NULL stratum as argument of dimension $n = d$ means that all the n-strata satisfying the adjacency/incidence condition expressed by the stratum arguments are to be returned; otherwise, if $n \neq d$ and the n-stratum argument is still NULL, no n-stratum imposes any adjacency/incidence restriction on the d-strata to be returned. In case $n = d$ and the n-stratum argument is not NULL, the operator $\blacktriangleright\!\!\triangleleft_d$ returns all the n-strata as before, except the n-stratum argument; if $n \neq d$ and the n-stratum is not NULL, the n-stratum imposes an additional adjacency/incidence restriction on the retrieved d-strata.

Let us consider again the mesh in Figure 5 to illustrate how the mask operator works in conjunction with the DiX data structure:

1. $\blacktriangleright\!\!\triangleleft_1(v_1,\text{NULL},\text{NULL}) = \{e_1, e_5\}$ directly returns all edges incident at v_1.
2. $\blacktriangleright\!\!\triangleleft_2(v_1,\text{NULL},\text{NULL}) = \{f_1, f_2\}$ indirectly returns all faces incident at v_1. This requires an intermediate call to $\blacktriangleright\!\!\triangleleft_1(v_1,\text{NULL},\text{NULL})$ to return all edges e_1, e_5 incident at v_1. Then, the operators $\blacktriangleright\!\!\triangleleft_2(\text{NULL}, e_1,\text{NULL})$ and $\blacktriangleright\!\!\triangleleft_2(\text{NULL}, e_5,\text{NULL})$ are called to compute those faces incident on e_1 and e_5.
3. $\blacktriangleright\!\!\triangleleft_0(\text{NULL}, e_1,\text{NULL}) = \{v_1\}$ directly returns its bounding vertices of e_1.
4. $\blacktriangleright\!\!\triangleleft_2(\text{NULL}, e_1,\text{NULL}) = \{f_1, f_2\}$ directly returns faces incident on e_1.
5. $\blacktriangleright\!\!\triangleleft_0(\text{NULL},\text{NULL}, f_1) = \{v_1\}$ indirectly returns all vertices bounding f_1. This requires an intermediate call to $\blacktriangleright\!\!\triangleleft_1(\text{NULL},\text{NULL}, f_1)$ to first determine all edges bounding f_1. Then, the operator $\blacktriangleright\!\!\triangleleft_0(\text{NULL}, e_i,\text{NULL})$ is called for each edge e_i in order to determine vertices bounding e_i and f_1.

The mask operator enables a fast and local management of stratifications in a way that there is no need to handle all the data structure constituents. Thus, the operational performance of the mask operator holds independently of the mesh size. This is very important for handling large meshes (e.g. the horse in Figure 4(c) has over 2×10^5 strata), in particular in real-time operations (e.g. local deformations) of meshes.

7. Data structure implementation

The DiX data structure represents a stratified object `Object` through a point set `PointSet` associated to a stratification `Complex` (or stratum complex). A point set can be the empty set, but it usually has at least a component `Component`. Every component is associated to a substratification or subcomplex `Subcomplex`.

The class `Complex` implements a n-dimensional stratification as a whole. A complex is represented by a family of subcomplexes, each one of which is associated to a component of its underlying point set. So, we can think of a hierarchy in which a complex contains subcomplexes, and each subcomplex is viewed as a set of strata. The DiX data structure is then as follows:

```
typedef vector<Stratum*> svector;

typedef vector<Subcomplex*> sxvector;
```

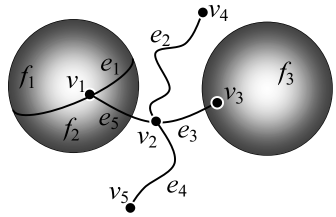

Figure 5: *A non-manifold stratified object.*

```
typedef vector<Component*> cvector;

typedef vector<Subcomplex*> hvector;

class Object {
  int         oid;      // object id
  PointSet    *ps;      // underlying pointset
  Complex     *cx;      // stratification
}

class Complex {
  int         cxid;     // stratification id
  Object      *o;       // object
  sxvector    sxv;      // subcomplexes
}

class Subcomplex {
  int         sxid;    // substratification id
  Complex     *sx;     // container complex
  svector     sxv;     // set of strata
}

class Stratum {
  int         sid;     // stratum id
  int         d;       // dimension
  svector     as;      // adjacent strata
  svector     is;      // incident strata
  PointSet    *ps;     // underlying pointset
  Vector      *vn;     // orientation
  hvector     *h;      // homotopy descriptor
  Subcomplex *sx;      // container subcomplex
}

class PointSet {
  int         psid;    // pointset id
  Geometry    *g;      // geometry descriptor
  cvector     cv;      // components
  Object      *o;      // container object
}

class Component {
  int         cid;     // component id
  Geometry    *g;      // geometry descriptor
  hvector     *h;      // homotopy descriptor
  PointSet    *ps;     // container pointset
  Subcomplex *sx;      // substratification
}
```

Note that the class `Stratum` implements a stratum as composed by a set of adjacent $(n-1)$-strata and a set of incident $(n+1)$-strata.

A `Geometry` is associated to each `PointSet`, `Component`, and `Stratum`. For example, the point set $(x-1)(x-2)=0$ has two components, $x-1=0$ and $x-2=0$, which are two parallel planes in $\mathbb{R}^3$. Therefore, the geometry of a point set distinguishes from the geometry of its components. Analogously, the geometry of a point set is not the same as the geometry of its associated strata. For example, the Cartan umbrella $x^2-zy^2=0$ as whole is a point set, but its associated Verdier strata have different geometric descriptions that result from the resolution of its singularities. The class `Geometry` is general enough to allow us to specialize point sets defined implicitly or parametrically.

Unlike most b-rep data structures, the DiX data structure is not topologically-oriented provided that it does not include any oriented strata. However, for rendering purposes in $\mathbb{R}^3$, it is geometrically-oriented by the vertex normal `vn` in the class `Stratum`. Each vertex normal is determined from the face normals around it.

The DiX data structure also includes a shape descriptor `hvector` for strata and components. Basically, the shape descriptor of a stratum is a set of tuples, each containing a set of subcomplexes for its boundary components –it may be boundary-incomplete–, holes, and frontier components. In a way, this shape descriptor is related to the Euler characteristic (1), while the shape descriptor for components has to do with the Euler characteristic (2). However, some applications such as, for example, multiresolution meshes need not these shape descriptors because they use cells, i.e. strata without holes.

8. Relation to other representations

In general terms, representations for stratifications can be classified into *implicit* and *explicit* representations. Implicit representations consist of a set of strata together with a set of functions to retrieve topological relations. For example, the cell-tuple structure[2] and the d-dimensional generalized maps[9] are implicit representations for d-dimensional subdivided manifolds and quasi-manifolds, respectively, in which strata are restricted to cells.

Explicit representations are based on the explicit encoding of strata plus some adjacency and incidence relations. They include edge-based representations[1, 13, 26, 8] and incidence graph-based representations[3, 27, 21]. Edge-based representations are usually restricted to $\mathbb{R}^3$, but their strata admit holes (with the exception of edges). Incidence graph-based representations are restricted to cell complexes in $\mathbb{R}^d$.

The DiX data structure extends the incidence graph-based representation to stratified objects in $\mathbb{R}^d$. These stratified objects need not be boundary-complete nor manifold. Besides, their strata and components have shape descriptors to ease the shape reasoning and editing of stratified objects.

9. Conclusions

One of the main problems in geometric modelling is the lack of general data structures for representing objects independently of dimension, manifoldness, and incomplete boundaries. Another problem of current geometric data structures is their known inability to cope with separate manipulation of subobjects as required by CAD systems, and feature-based modellers in particular. The result is a poor abstraction and design of geometric modellers and proliferation of *ad hoc* solutions, say external data structures, for new problems, with consequent difficulties in software maintenance.

The DiX data structure has been designed from a general mathematical theory involving different ways to look at the shape of a point set. Consequently, the geometry-topology traditional view of a b-rep data structure has given place to a geometric kernel in which we need to have other shape descriptors, e.g. homotopy or hole descriptors. A hole may need be represented by a set of strata, i.e. a subcomplex. But, a subcomplex may be also used as stratified subobjects for geometric features, boolean primitives, or even user-defined subobjects.

The conciseness of the DiX comes from fact that it does not include oriented strata. These oriented strata (e.g. halfedges, coedges, face-uses, etc.) are usual in b-rep data structures, but are extremely verbose. The orientation of the DiX is geometric, though it is computed by inducing a topological orientation on the frontier of each stratum, what is done with the help of the mask operator. Besides, DiX belongs to the optimal class $C_{2n}^{(n+1)^2}$ of geometric data structures.

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