Supplementary Document A
Geometry Modeling Using Focal Surfaces

1 Proof to Focal Surface Property 1 and 2

To prove the normal tangentiality, we calculate the tangent vector along the first principal direction f_1 on the first focal surface Γ_1:

$$D_{\vec{f}_1}\Gamma_1 = D_{\vec{f}_1}S - \frac{1}{K_1}D_{\vec{f}_1}\vec{N} + \frac{1}{K_1^2}D_{\vec{f}_1}(K_1)\vec{N}$$
$$= \vec{f}_1 - \frac{1}{K_1}K_1\vec{f}_1 + \frac{1}{K_1^2}D_{\vec{f}_1}(K_1)\vec{N} = \frac{1}{K_1^2}D_{\vec{f}_1}(K_1)\vec{N} \quad (1)$$

Therefore, N coincides with the tangent direction of Γ_1 along f_1. Similarly, N is also tangent to Γ_2.

Next, we compute the tangent vector along the second principal direction f_2 on Γ_1.

$$D_{\vec{f}_2}\Gamma_1 = D_{\vec{f}_2}S - \frac{1}{K_1}D_{\vec{f}_2}\vec{N} + \frac{1}{K_1^2}D_{\vec{f}_2}(K_1)\vec{N} \quad (2)$$
$$= \vec{f}_2 - \frac{1}{K_1}K_2\vec{f}_2 + \frac{1}{K_1^2}D_{\vec{f}_2}(K_1)\vec{N}$$

This means the second tangent direction on Γ_1 is a linear combination of the surface normal $\vec{N}$ and the second principal direction $\vec{f}_2$. Since $\vec{N}$ is tangent to Γ_1, then $\vec{f}_2$ is also tangent to Γ_1. This provides a second property of the focal surface: that the second principal direction rules the first focal surface, and the first principal direction rules the second focal surface. Furthermore, we can find the normal of Γ_1 as the cross product of $D_{\vec{f}_1}\Gamma_1$ and $D_{\vec{f}_2}\Gamma_1$. The normal of Γ_1 corresponds to $\vec{f}_1$.

2 The slit Direction and the Principal Directions

Here we show the two slits obtained in Section 4 have the same direction as the two principal directions on the original surface.

First, we consider the direction of the two slits. Recall that the normal ray characteristic equation picks three rays $r_1 = r$, $r_2 = r + r_x$ and $r_3 = r + r_y$ and finds the depth λ where they form a slit. Therefore, we can choose the three points P_1, P_2 and P_3 along r_1, r_2, and r_3 at $z = \lambda$ and compute their slope.

$$\begin{aligned} P_1 &= [u+\lambda\sigma, v+\lambda\tau, \lambda] \quad (3)\\ P_2 &= [(u+u_x)+\lambda(\sigma+\sigma_x), (v+v_x)+\lambda(\tau+\tau_x), \lambda]\\ P_3 &= [(u+u_y)+\lambda(\sigma+\sigma_y), (v+v_y)+\lambda(\tau+\tau_y), \lambda] \end{aligned}$$

Since P_1, P_2, and P_3 must lie on a line at λ, we only need to pick any two of them form a slit. Without loss of generality, assume P_1 and P_2 do not coincide, the direction of the first slit can be computed as

$$\vec{d} = P_2 - P_1 = [u_x + \lambda\sigma_x, v_x + \lambda\tau_x, 0] \quad (4)$$

First, we show slit P_1P_2 rule the focal surface. To do so, compute the two tangent directions $\vec{\Gamma}_x$ and $\vec{\Gamma}_y$ of the focal surface. I show $\vec{d}$, $\vec{\Gamma}_x$, and $\vec{\Gamma}_y$ are co-planar. It is easy to verify

$$Det[\vec{\Gamma}_x^{\,T}, \vec{\Gamma}_{y2}^{\,T}, \vec{d}^T] = \begin{vmatrix} u_x+\lambda\sigma_x & v_x+\lambda\tau_x & 0\\ u_x+\lambda_x\sigma+\lambda\sigma_x & v_x+\lambda_x\tau+\lambda\tau_x & \lambda_x\\ u_y+\lambda_y\sigma+\lambda\sigma_y & v_y+\lambda_y\tau+\lambda\tau_y & \lambda_y \end{vmatrix} = 0 \quad (5)$$

Therefore, the slit lies on the tangent plane of the caustic surface and hence it is tangent to the caustic surface.

Next, we show the two slits must be perpendicular to each other. To compute the direction of each slit, we assume the two slits lie at depth λ_1 and λ_2. Suppose the three rays intersect $z = \lambda_1$ plane at P_1, P_2 and P_3 and intersect λ_2 at Q_1, Q_2 and Q_3. We can compute the two direction of the slits as

$$\vec{d}_1 = P_2 - P_1 = [u_x + \lambda_1\sigma_x, v_x + \lambda_1\tau_x, 0] \quad (6)$$

Similarly, the direction of the second slit as

$$\vec{d}_2 = Q_2 - Q_1 = [u_x + \lambda_2\sigma_x, v_x + \lambda_2\tau_x, 0] \quad (7)$$

Finally, we show $\vec{d}_1 \cdot \vec{d}_2 = 0$. Since we model the surface as a local Monge patch, therefore, we have

$$\begin{aligned} [\sigma, \tau, 1] &= [-z_x, -z_y, 1]\\ [u, v, 0] &= \dot{P} - z\vec{D} = [x - zz_x, y - zz_y, 0] \quad (8) \end{aligned}$$

At the point we analyze, we have $x = y = z_x = z_y = 0$. Furthermore, we can compute the Gaussian and the Mean curvature as:

$$\begin{aligned} K &= \frac{z_{xx}z_{yy} - z_{xy}^2}{(1+z_x^2+z_y^2)^2} = z_{xx}z_{yy} - z_{xy}^2\\ 2H &= \frac{(1+z_x^2)z_{yy} + (1+z_y^2)z_{xx} - 2z_xz_yz_{xy}}{(1+z_x^2+z_y^2)^{\frac{3}{2}}} = z_{xx} + z_{yy} \quad (9) \end{aligned}$$

Substitute σ, τ, u, v with equation 8, we get

$$\vec{d}_1 \cdot \vec{d}_2 = (1 - \lambda_1 z_{xx})(1 - \lambda_2 z_{xx}) + \lambda_1\lambda_2 z_{xy}^2 \quad (10)$$

Furthermore, since $\lambda_1 = -\frac{1}{\kappa_1}$ and $\lambda_2 = -\frac{1}{\kappa_2}$, we have

$$\begin{aligned} \lambda_1 + \lambda_2 &= \frac{1}{\kappa_1} + \frac{1}{\kappa_2} = \frac{2H}{K} = \frac{z_{xx}+z_{yy}}{z_{xx}z_{yy} - z_{xy}^2} \quad (11)\\ \lambda_1\lambda_2 &= \frac{1}{\kappa_1\kappa_2} = \frac{1}{K} = \frac{1}{z_{xx}z_{yy} - z_{xy}^2} \end{aligned}$$

Substituting equation 11 into equation 10 gives $\vec{d}_1 \cdot \vec{d}_2 = 0$. Thus, the two slits must be perpendicular to each other.

Since the normal is also tangent to the focal surfaces, the directions of the two slits and the normal direction at the base surface must form a frame. Furthermore, the normal of the focal surfaces correspond to the principal directions, thus, the directions of the two slits correspond to the two principal directions.

3 Derivation of the Error Metric in Section 5

In this section we will show the metric for measuring the distance between the surrounding normal rays and the slits, as well as the derivation of the formula.

3.1 The Least Square Metric

Suppose we are given a central normal ray γ, its surrounding normal rays γ_i and their 2PP coordinates $[\sigma_i, \tau_i, u_i, v_i]$. We hope to estimate the two slits by minimizing the distance between the given surrounding normal rays and the slits in a least square manner.

In particular, we are looking for the optimal $[k_1, k_2, \theta]$ which minimizes certain objective function as explained in the following.

3.1.1 The Unknowns

- k_1*:* max principal curvature; equals $\frac{1}{-z_1}$, where z_1 is the depth of focal surface Γ_1.
- k_2*:* min principal curvature; equals $\frac{1}{-z_2}$, where z_1 is the depth of focal surface Γ_2.
- θ*:* the angle between the $\mathbf{X}$ axis and $slit1$ on the $\mathbf{XY}$ plane (in the tangent coordinate system, as explained in below).

3.1.2 Objective Function

The error metric we want to minimize is a least square function:

$$\sum_{\gamma_i \in 1-ring-neighbor(\gamma)} D^2(\gamma_i, l_1) + D^2(\gamma_i, l_2)$$

where

$$D(\gamma_i, l_1) = (u_i k_1 - \sigma_i)\cos(\theta) + (v_i k_1 - \tau_i)\sin(\theta)$$

$$D(\gamma_i, l_2) = -(u_i k_2 - \sigma_i)\sin(\theta) + (v_i k_2 - \tau_i)\cos(\theta)$$

3.2 The Derivation

In this part, we outline the derivation of the least square formula mentioned above. First, we introduce two coordinate systems used in the derivation; then we explain how the angle difference is used to measure the distance between surrounding normal rays and the slits.

3.2.1 Tangent Coordinate System and Slit Coordinate System

Given a point P on the surface, the *tangent coordinate system* at this point takes P as the origin, P's normal $\overrightarrow{n}$ as $\mathbf{Z}$ axis, and two orthogonal directions on the tangent plane as $\mathbf{X}$ and $\mathbf{Y}$ axis.

In this tangent coordinate system, the uv plane and st plane are located at $z = 0$ and $z = 1$ respectively, with u and s taking the X direction, v and t the Y direction. Suppose γ_i is one of the surrounding normal rays with 2PP coordinate $[\sigma, \tau, u, v]$; it intersects uv and st plane at $[u, v, 0]$ and $[s, t, 1]$, where

$$s = u + \sigma$$

$$t = v + \tau$$

We can get the *slit coordinate system* $\mathbf{L}_1\mathbf{L}_2\mathbf{Z}$ by projecting the two slits onto $\mathbf{XY}$ plane, where L_1 and L_2 are the images of $slit_1$ and $slit_2$ on $\mathbf{XY}$ plane. Suppose the angle between X to L_1 is θ, we can rotate the $\mathbf{XYZ}$ coordinate system around $\mathbf{Z}$ axis by angle θ to get the $\mathbf{L}_1\mathbf{L}_2\mathbf{Z}$. The Jaccobi of the transformation is

$$J = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotating uv and st plane around Z by angle θ, we get $u'v'$ plane and $s't'$ plane. And each normal ray γ_i intersects $u'v'$ and $s't'$ plane at $[u', v', 0]$ and $[s', t', 1]$ in the tangent coordinate system. Then we have the following equations:

$$\begin{pmatrix} u' \\ v' \\ 0 \end{pmatrix} = J \begin{pmatrix} u \\ v \\ 0 \end{pmatrix}, \begin{pmatrix} s' \\ t' \\ 1 \end{pmatrix} = J \begin{pmatrix} s \\ t \\ 1 \end{pmatrix}$$

Then we have

$$\begin{aligned} u' &= u\cos(\theta) + v\sin(\theta) \\ v' &= -u\sin(\theta) + v\cos(\theta) \\ s' &= s\cos(\theta) + t\sin(\theta) \\ t' &= -s\sin(\theta) + t\cos(\theta) \end{aligned}$$

3.2.2 Evaluating Distance by Angle Difference

Let $\gamma_i^{L_1}$ and $\gamma_i^{L_2}$ be the images of γ_i on $\mathbf{L}_1\mathbf{Z}$ plane and $\mathbf{L}_2\mathbf{Z}$ plane respectively. If γ_i is close enough to γ, then $\gamma_i^{L_1}$ and $\gamma_i^{L_2}$ should be close to focal point $Z_1(0,0,z_1)$ and $Z_2(0,0,z_2)$ respectively.

First, let's build a formula to measure the "closeness" between $\gamma_i^{L_1}$ and Z_1 on $\mathbf{L}_1\mathbf{Z}$ plane. On this plane, $\gamma_i^{L_1}$ intersects $u'v'$ and $s't'$ at $U(u', 0, 0)$ and $S(s', 0, 1)$ respectively. Let α_1 be the angle between $\overline{US}$ (i.e. $\gamma_i^{L_1}$) and axis $\mathbf{L}_1$, α_2 be the angle between $\overline{Z_1 U}$ and axis $\mathbf{L}_1$. It follows that

$$\cot(\alpha_1) = \frac{(s' - u')}{1 - 0} = s' - u'$$

$$\cot(\alpha_2) = \frac{u'}{0 - z_1} = u' k_1$$

If $\gamma_i^{L_1}$ is very close to Z_1, the difference between $\cot\alpha_1$ and $\cot\alpha_2$ should be close to zero:

$$\begin{aligned} D(\gamma_i, l_1) &= u' k_1 - (s' - u') \\ &= (uk_1 - \sigma)\cos(\theta) + (vk_1 - \tau)\sin(\theta) \end{aligned}$$

Similarly, we can derive another formula to measure the "closeness" between $\gamma_i^{L_2}$ and Z_2 on $\mathbf{L}_2\mathbf{Z}$ plane:

$$\begin{aligned} D(\gamma_i, l_2) &= v'k_2 - (t' - v') \\ &= -(uk_2 - \sigma)\sin(\theta) + (vk_2 - \tau)\cos(\theta) \end{aligned}$$

Finally, in order to estimate the position of the two slits, we just need to minimize the above angle difference upon all the normal rays γ_i surrounding the central normal ray γ:

$$\sum_{\gamma_i \in 1-ring-neighbor(\gamma)} D^2(\gamma_i, l_1) + D^2(\gamma_i, l_2)$$

This completes the derivation.