Reverse Diffusion for Smooth Buildup of Progressivly Transmitted Geometry†

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Abstract

In this paper we consider 3D object surfaces which can be represented as scalar functions defined on the sphere. These objects can be modeled as series of spherical harmonic functions. A simple progressive transmission scheme could be implemented which transmits the expansion coefficients one by one and thus implements a coarse to fine reconstruction. The buildup of the object according to this scheme is not completely smooth: Wavy patterns appear which disappear in subsequent stages and are replaced by finer spurious patterns and so on. We propose a remedy for this behavior which is based on the simulation of a reversed diffusion process on the sphere.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Curve, surface, solid, and object representations

1. Introduction

Compression of 3D surface data (*geometry compression*) and especially schemes which allow for progressive transmission of such compressed data are of crucial importance for the efficient transmission of 3D surfaces, e.g. over the Internet. Solutions have been developed, based mainly on surfaces represented as triangular meshes [13, 8, 10, 9].

In this paper we will consider a different surface model which is valid for all surfaces which can be described as functions on the sphere. These functions can be expressed as series of spherical harmonic functions which corresponds to the Fourier expansion of a function on the plane.

In earlier work the favorable properties of spherical harmonics (SH's) (orthonormality, completeness, coarse-to-fine hierarchy) have been exploited for the representation of 3D object surfaces. The use of spherical harmonics was first proposed by Schudy and Ballard [12, 1]. They model the dynamic heart volume by SH's with periodically time-varying coefficients. Later SH's have been used to compress coarse scale head models [7]. SH's can be combined with spherical Gabor filters in order to take care of fine detail locally [3] or with other surface harmonics like cylindrical or elliptical harmonics to account for global shape appropriately [11].

A simple progressive transmission scheme can be imagined that transmits the spherical harmonic coefficients of a given surface starting with the low frequency components and continuing with increasingly fine detail (higher frequency). However, as we will show, this leads to the intermediate introduction of wavy patterns in smooth areas of the object. These artefacts are due to the global support of spherical harmonics and vanish as higher frequency components are added. In this paper we propose a method for blending in the coefficients which leads to a smooth buildup of the object without intermediate artefacts.

Recently, it has been shown that for surfaces defined as functions on the sphere a linear scale space can be introduced via the diffusion equation on the sphere [2]. The analogy between the Fourier transform in the image plane and the spherical harmonic expansion carries quite far: The spherical diffusion equation can easily be solved in the spectral

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domain. Furthermore the Green's function of the spherical diffusion equation can be interpreted as the spherical Gaussian. Linear diffusion on the sphere can thus be preformed by convolution with a spherical Gaussian (smoothing) filter. The method proposed in this paper is based on the reversion of this diffusion process.

The structure of this paper is as follows. In Sect. 2 the necessary mathematical tools will be provided. In Sect. 3 we briefly outline the surface smoothing process based on spherical diffusion. The smooth buildup scheme based on reverse diffusion will be described in Sect. 4 before we close with a conclusion in Sect. 5.

2. Mathematical Preliminaries

We will use the standard spherical coordinates to parameterize the unit sphere

$$\mathbb{S}^2 = \left\{ \eta(\varphi,\vartheta) := \begin{pmatrix} \cos(\varphi)\sin(\vartheta) \\ \sin(\varphi)\sin(\vartheta) \\ \cos(\vartheta) \end{pmatrix} \right\}, \tag{1}$$

with $\varphi \in [0,2\pi), \vartheta \in [0,\pi]$. The spherical harmonic functions $Y_{lm} : \mathbb{S}^2 \to \mathbb{C}$ are defined as the everywhere regular eigenfunctions of the spherical Laplace operator [5]. These functions constitute a complete orthonormal system of the space of square integrable functions on the sphere $L^2(\mathbb{S}^2)$. In spherical coordinates the Y_{lm} are given by

$$Y_{lm}(\eta) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos(\vartheta))e^{im\varphi}, \tag{2}$$

with $l \in \mathbb{N}$ and $|m| \leq l$. Here P_l^m denote the associated Legendre polynomials [4]. Spherical harmonics are orthogonal

$$\int_{\mathbb{S}^2} Y_{lm}(\eta)Y^*_{l'm'}(\eta)\,d\eta = \delta_{ll'}\delta_{mm'}. \tag{3}$$

and complete in $L^2(\mathbb{S}^2)$ such that any $f \in L^2(\mathbb{S}^2)$ can be expanded into spherical harmonics:

$$f = \sum_{l\in\mathbb{N}}\sum_{|m|\leq l} \hat{f}_{lm}Y_{lm} \quad \text{with} \quad \hat{f}_{lm} = \int_{\mathbb{S}^2} f(\eta)\,Y^*_{lm}(\eta)\,d\eta, \tag{4}$$

where $\cdot^*$ denotes complex conjugation. For the surface element on the sphere we use the shorthand notation $d\eta := \sin(\vartheta)\,d\vartheta\,d\varphi$. The set of coefficients $\hat{f}_{lm}$ is called the *spherical Fourier transform* or the *spectrum* of f.

Spherical harmonics are eigenfunctions of the Laplace operator restricted to the sphere $\Delta_{\mathbb{S}^2}$ as well as of the derivative operator with respect to the azimuthal angle φ:

$$\Delta_{\mathbb{S}^2}Y_{lm} = -l(l+1)Y_{lm}, \quad \partial_\varphi Y_{lm} = imY_{lm}. \tag{5}$$

We are going to apply this filter to a given surface by correlation. We make use of the following theorem. For functions $f,h \in L^2(\mathbb{S}^2)$ with $\hat{h}_{lm} = 0$ for $m \neq 0$ (i.e. for a *rotationally symmetric filter* h) the spectrum of the correlation is a pointwise product of the spectra of f and h

$$(\widehat{f \star h})_{lm} = \sqrt{\frac{4\pi}{2l+1}}\hat{f}_{lm}\hat{h}_{l0}. \tag{6}$$

A similar result starting from another definition of convolution has been proven by Driscoll and Healy [6].

3. Spherical Diffusion Smoothing

Using (4) and (5) it can be easily verified that the spherical function G given by its spectrum as

$$\widehat{G(\cdot;t)}_{lm} = \begin{cases} \sqrt{\frac{2l+1}{4\pi}}e^{-l(l+1)t} & \text{if } m=0 \\ 0 & \text{else .} \end{cases} \tag{7}$$

solves the spherical diffusion equation $\Delta_{\mathbb{S}^2}u = \partial_t u$. The function G is known as the Gauss-Weierstrass kernel or the spherical Gaussian function. A derivation of this result can be found in [2]. Combination of (7) and (6) shows that the coefficients of the smoothed surface can be obtained from the original coefficients by

$$\hat{f}_{lm} \mapsto \hat{f}_{lm}e^{-l(l+1)t} =: (\hat{f}^t)_{lm}. \tag{8}$$

Here, f^t is the result of linear spherical diffusion applied to the function f with diffusion time t. Figure 1 shows an example of an object undergoing a spherical diffusion process.

4. Reverse Diffusion

Assume a progressive transmission of 3D surface data is performed by transmission of spherical harmonic coefficients $\hat{f}_{lm}$. First the coefficient $\hat{f}_{00}$ with $l=0$ is transmitted, followed by the three coefficients with $l=1$ and so on. Due to the observation that the index l can be considered as a frequency, the described scheme transmits the surface data in a coarse to fine manner. An example for a different number of transmitted coefficients is shown in Fig. 2. We have not applied any quantization scheme to the coefficients nor any subsequent coding. If the coefficients are transmitted at 4 Bytes per coefficient the reconstructions shown in Fig. 2 require $4L^2$ Bytes, i.e. 14400 Bytes for the most detailed level Fig. 2(f). These reconstructions can be considered as results of ideal bandpass filtering of the original object.

Figure 2 reveals an undesirable effect of this reconstruction method. Spherical harmonic functions are of global support. Thus, adding fine detail in some places leads to ringing effect in smooth regions of the object (see e.g. the top of the head in Fig. 2(c)). This wavy appearance is only later canceled out as more coefficients are added. Thus, spurious detail is introduced at some stages which vanishes later in the reconstruction process.

It would be visually much more pleasing to build up the object as shown in Fig. 1 but in reverse order. This would

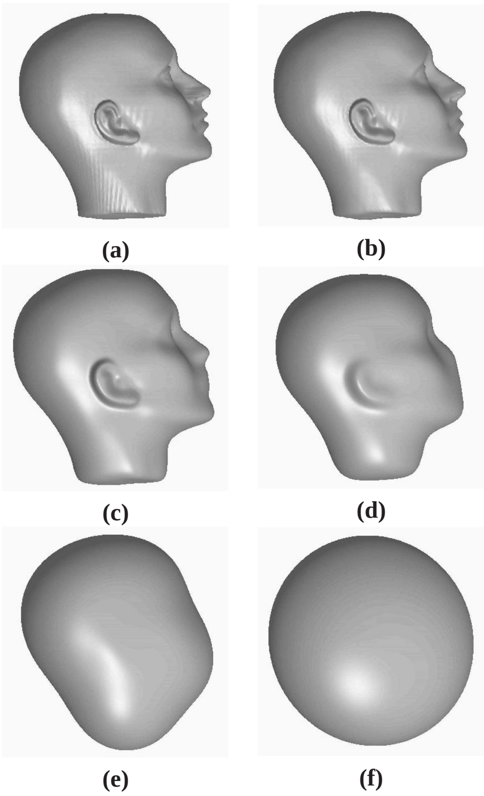

Figure 1: ***(a)*** *Original data.* ***(b)*** *Smoothed with* $t = 10^{-4}$. ***(c)*** $t = 10^{-3}$. ***(d)*** $t = 5 \cdot 10^{-3}$. ***(e)*** $t = 0.03$. ***(f)*** $t = 0.3$.

correspond to a reverse diffusion process. The scale space properties of the hierarchy generated by spherical diffusion smoothing would guarantee that no spurious detail is introduced at any stage of the reconstruction process.

Equation (8) shows that any coefficient $\hat{f}_{lm}$ of the original function f will contribute to $\hat{f}^t{}_{lm}$ for any given $t \geq 0$

$$\hat{f}_{lm} \neq 0 \Rightarrow \hat{f}^t{}_{lm} \neq 0 \quad \forall t \geq 0. \tag{9}$$

Thus, in order to build up f via reverse diffusion we would need to wait for all coefficients to be available before even the very coarse structure could be expressed. This clearly contradicts our aim to use each incoming coefficient to improve the reconstruction.

There is however still hope if we settle for a compromise. Assume we want to construct the smoothed version

$$f^t = \sum_{lm} \hat{f}_{lm} Y_{lm} e^{-l(l+1)t}. \tag{10}$$

Since this requires knowledge of all coefficients $\hat{f}_{lm}$ we decide to approximate f^t by the finite sum containing only those coefficients which have not been strongly attenuated. To be concrete, we only use the coefficients $\hat{f}_{lm}$ with l such

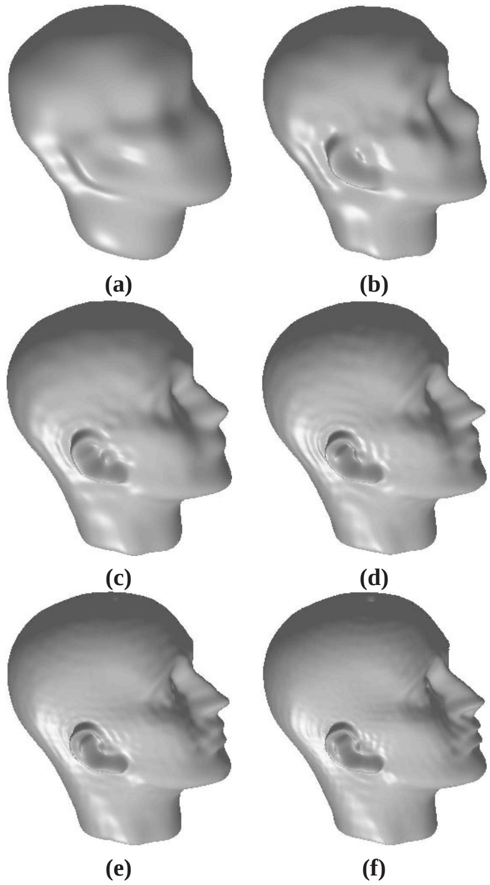

Figure 2: *Progressive reconstruction of an object from the first* L^2 *spherical harmonic coefficients.* ***(a)*** $L = 10$. ***(b)*** $L = 20$. ***(c)*** $L = 30$. ***(d)*** $L = 40$. ***(e)*** $L = 50$. ***(f)*** $L = 60$.

that

$$e^{-l(l+1)t} \geq \varepsilon, \quad \varepsilon \in (0, 1). \tag{11}$$

This will still lead to a ringing effect, but much less so than in the case of the ideal low-pass results shown in Fig. 2.

Assume we have received all coefficients up to $l = L$. For a given ε we determine $t = t_{\varepsilon,L}$ such that (11) is fulfilled as equality for $l = L$

$$t_{\varepsilon,L} = -\frac{\ln(\varepsilon)}{L(L+1)}. \tag{12}$$

Thus, we propose, given coefficients $\hat{f}_{lm}$ up to $l = L$, to reconstruct

$$f_L^{\varepsilon} = \sum_{l=0}^{L} \sum_{|m| \leq l} \hat{f}_{lm} Y_{lm} e^{-l(l+1)t_{\varepsilon,L}}. \tag{13}$$

Figures 3 - 5 show results for different values of ε.

It can be seen that building the reconstruction up by reverse diffusion results in a smoother development. However,

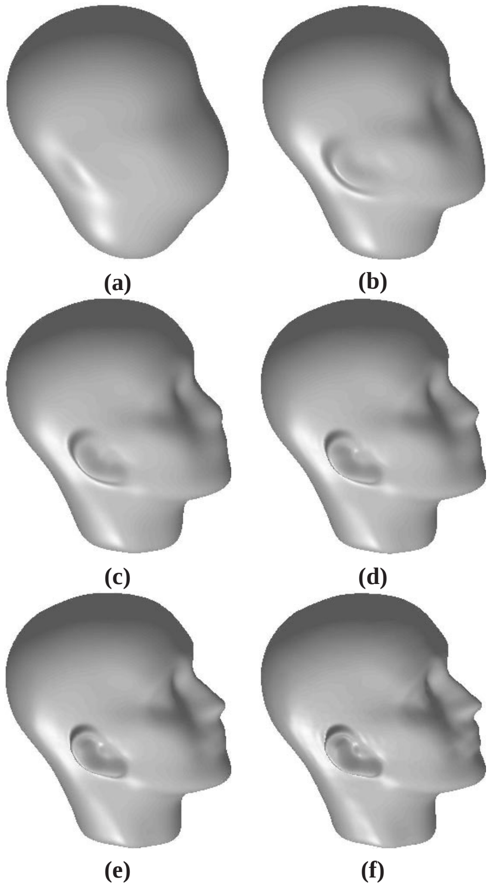

Figure 3: *Progressive reconstruction from the first* L^2 *spherical harmonic coefficients using (13) with* $\varepsilon = 0.1$. ***(a)*** $L = 10$. ***(b)*** $L = 20$. ***(c)*** $L = 30$. ***(d)*** $L = 40$. ***(e)*** $L = 50$. ***(f)*** $L = 60$.

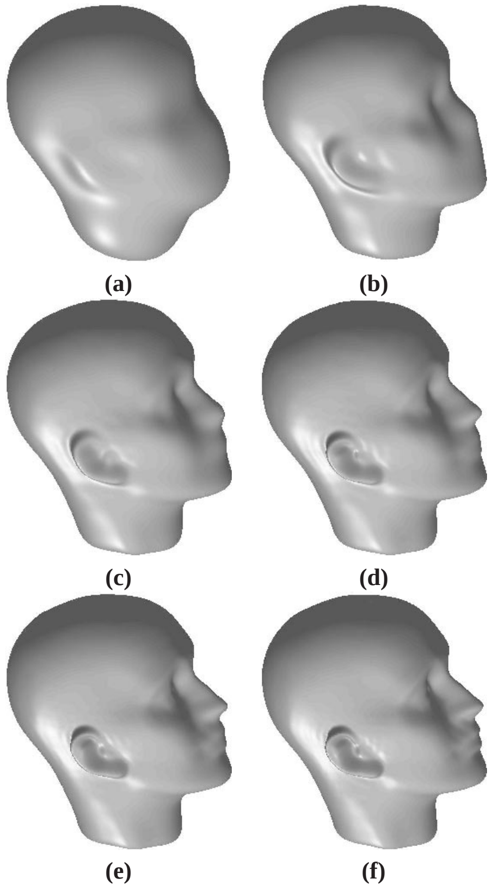

Figure 4: *Progressive reconstruction from the first* L^2 *spherical harmonic coefficients using (13) with* $\varepsilon = 0.3$. ***(a)*** $L = 10$. ***(b)*** $L = 20$. ***(c)*** $L = 30$. ***(d)*** $L = 40$. ***(e)*** $L = 50$. ***(f)*** $L = 60$.

this has to be paid for by a slower appearance of fine detail. In a practical application situation the parameter ε can be chosen according to the preferences. A small ε will lead to a very smooth buildup, very close to actual reverse diffusion whereas $\varepsilon = 1$ corresponds to taking each received coefficient immediately into account in full.

5. Conclusion

In this paper we have considered a progressive geometry transmission scheme based on the transmission of a series of spherical harmonic expansion coefficients. Just adding each new incoming coefficient to the reconstruction yields undesirable ringing effects. We have proposed a method which is based on the approximation of a reverse diffusion process. This leads to a "blending in" of the newly received expansion coefficients which in result in a smooth buildup of the reconstruction.

Future work will have to concentrate on the development of an actual compression scheme based on spherical harmonic coefficients to make this means of progressive transmission competitive.

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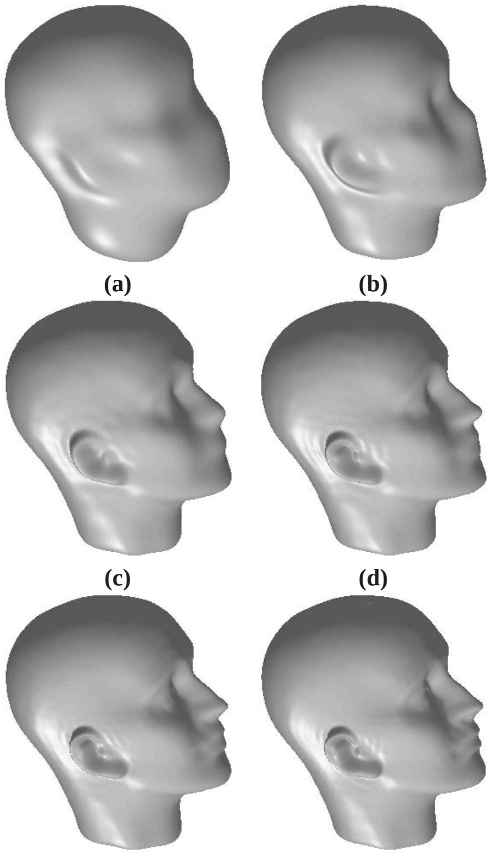

Figure 5: *Progressive reconstruction from the first L^2 spherical harmonic coefficients using (13) with $\varepsilon = 0.5$.* ***(a)*** $L = 10$. ***(b)*** $L = 20$. ***(c)*** $L = 30$. ***(d)*** $L = 40$. ***(e)*** $L = 50$. ***(f)*** $L = 60$.

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