Tutorial

Inverse Spectral Geometry

2/4

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EUROGRAPHICS 2021

Outline

- Laplacian eigenvalues and eigenvectors
- Fourier analysis on manifolds
- Functional maps
- Shape difference operators
- Shape-from-operator inverse problems

Entdeckungen

über die

Theorie des Klanges

von

Ernst Florens Friedrich Chladni,

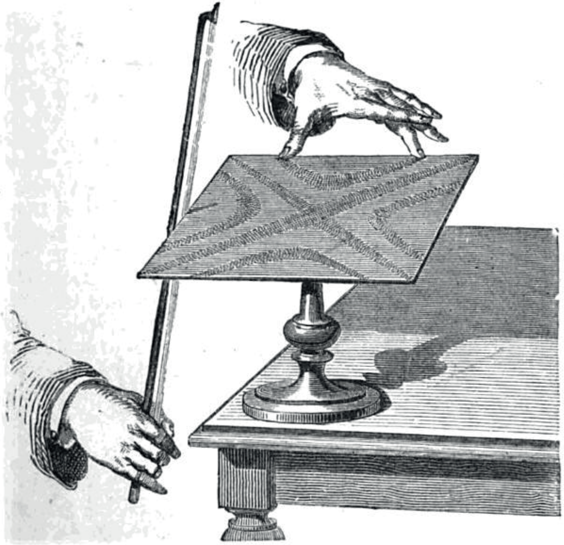
der Philosophie und Rechte Doctor zu Wittenberg.

Mit eilf Kupfertafeln.

Leipzig,

bey Weidmanns Erben und Reich.

1787.



Ernst Chladni (1756-1827)

Interpretation of Chladni plates

Behavior of waves on the plate is guided by the wave equation

$$f_{tt}(x,t) = -\Delta f(x,t)$$

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Plugging into the equation

$$\varphi(x)\tau_{tt}(t) = -\Delta\varphi(x)\tau(t)$$

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$$\frac{\tau_{tt}(t)}{\tau(t)} = -\frac{\Delta\varphi(x)}{\varphi(x)} = -\lambda$$

Spatial part of the solution satisfies the Helmholtz equation

$$\Delta\varphi(x) = \lambda\varphi(x)$$

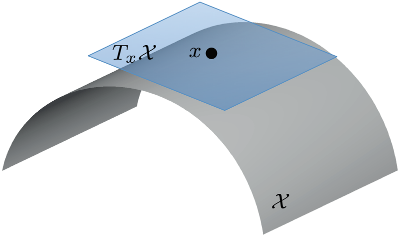
λ = vibration frequencies

$$\lambda = \text{vibration frequencies}$$

$$\phi = \text{vibration modes}$$

Riemannian geometry in one minute

- Manifold $\mathcal{X}$ = topological space
- No global Euclidean structure
- Tangent plane $T_x\mathcal{X}$ = local Euclidean representation of manifold $\mathcal{X}$ around x



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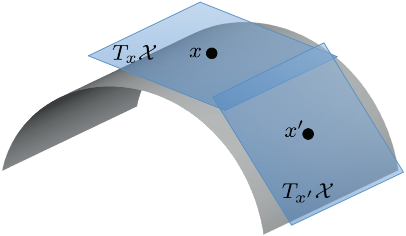
$$\langle \cdot, \cdot \rangle_x : T_x\mathcal{X} \times T_x\mathcal{X} \to \mathbb{R}$$

depending smoothly on x



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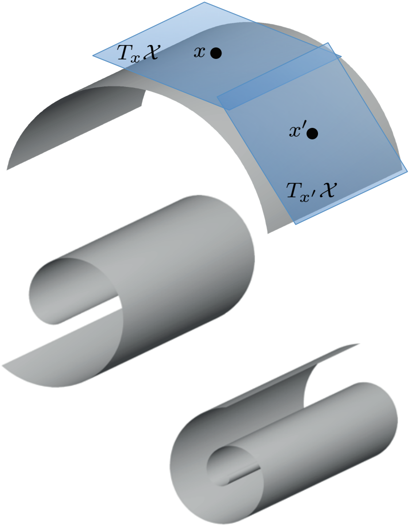
Isometry = metric-preserving shape deformation



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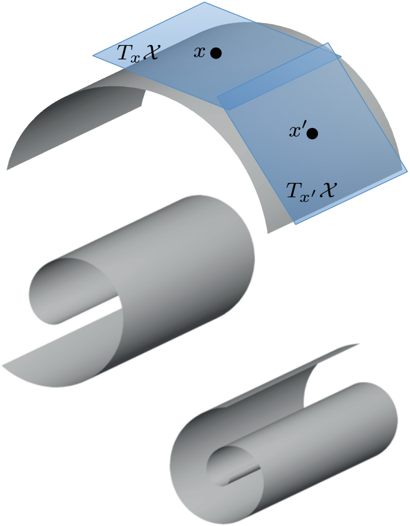
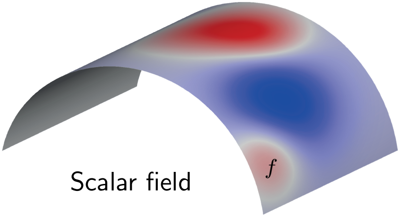
Isometry = metric-preserving shape deformation

Intrinsic = expressible solely in terms of Riemannian metric



Calculus on manifolds: scalar and vector fields

- Scalar field $f : \mathcal{X} \to \mathbb{R}$



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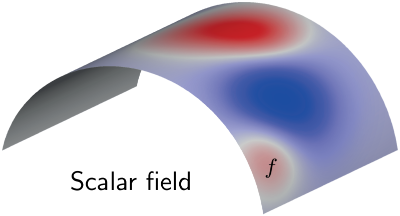
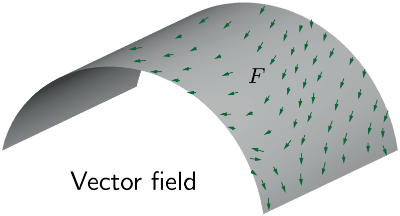
- Vector field $F : \mathcal{X} \to T\mathcal{X}$





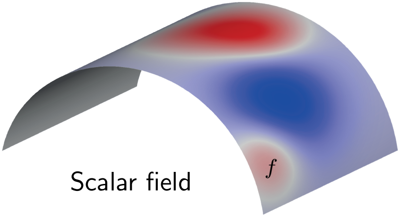
Calculus on manifolds: scalar and vector fields

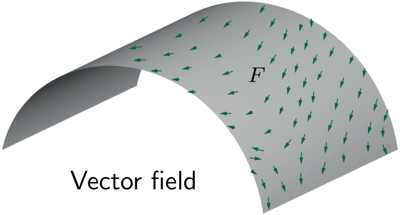
- Scalar field $f : \mathcal{X} \to \mathbb{R}$
- Vector field $F : \mathcal{X} \to T\mathcal{X}$
- Hilbert space with inner products

$$\langle f, g \rangle_{\mathcal{F}(\mathcal{X})} = \int_{\mathcal{X}} f(x) g(x) \mathrm{d}x$$

$$\langle F, G \rangle_{\mathcal{F}(T\mathcal{X})} = \int_{\mathcal{X}} \langle F(x), G(x) \rangle_x \mathrm{d}x$$

where $\mathrm{d}x =$ area element induced by the Riemannian metric

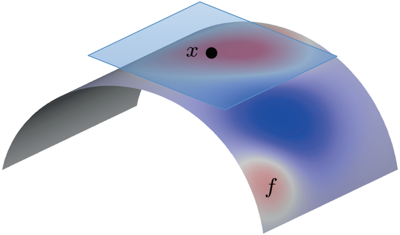




Calculus on manifolds: gradient

- Differential $\mathrm{d}f : T\mathcal{X} \to \mathbb{R}$ acting on vector fields

$$\mathrm{d}f(x) = \langle \nabla f(x), \cdot \rangle_x$$



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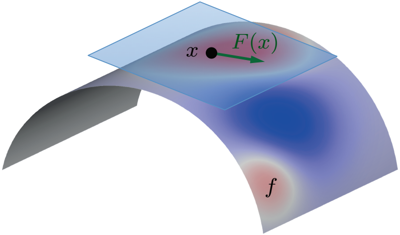
- Directional derivative

$$\mathrm{d}f(x)F(x) = \langle \nabla f(x), F(x) \rangle_{T_x\mathcal{X}}$$

"how much f changes at x in direction $F(x)$"



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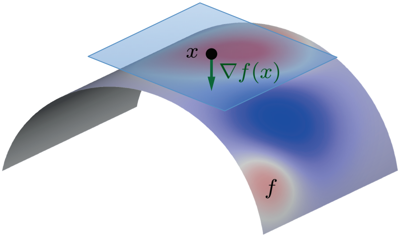

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"how much f changes at x in direction $F(x)$"

- Intrinsic gradient operator

$$\nabla f : \mathcal{F}(\mathcal{X}) \to \mathcal{F}(T\mathcal{X})$$

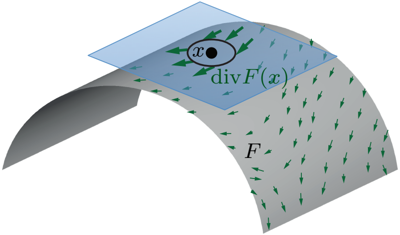
"direction of steepest change of f"

Calculus on manifolds: divergence

- Intrinsic divergence operator

$$\mathrm{div} : \mathcal{F}(T\mathcal{X}) \to \mathcal{F}(\mathcal{X})$$

"net flow of field F at x"



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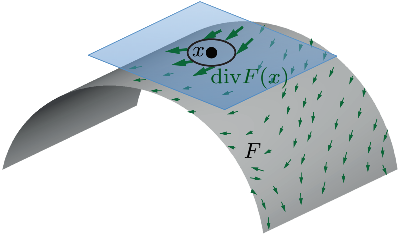
  Formal adjoint of the gradient

  $$\langle F, \nabla f \rangle_{\mathcal{F}(T\mathcal{X})} = \langle \nabla^* F, f \rangle_{\mathcal{F}(\mathcal{X})} = \langle -\mathrm{div} F, f \rangle_{\mathcal{F}(\mathcal{X})}$$

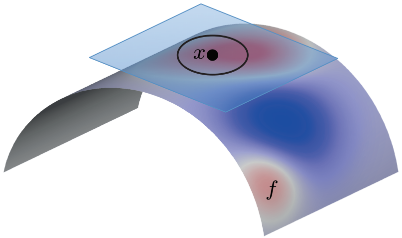


Calculus on manifolds: Laplacian

- Laplacian $\Delta : \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{X})$

$$\Delta f = -\mathrm{div}(\nabla f)$$

"difference between $f(x)$ and average value of f around x"



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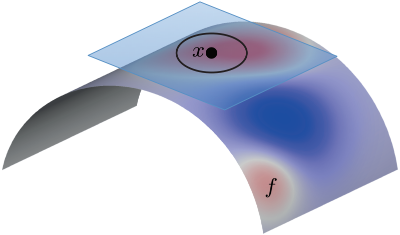

- Intrinsic (expressed solely in terms of the Riemannian metric)

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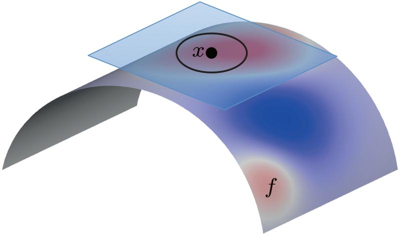

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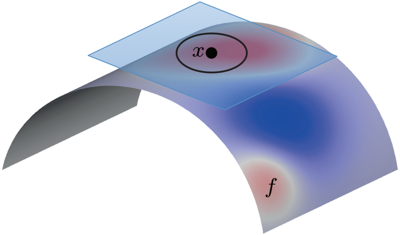

- Intrinsic (expressed solely in terms of the Riemannian metric)
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- Self-adjoint $\langle \Delta f, g \rangle_{\mathcal{F}(\mathcal{X})} = \langle f, \Delta g \rangle_{\mathcal{F}(\mathcal{X})}$

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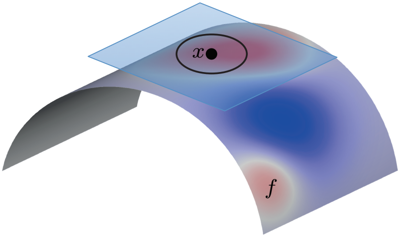

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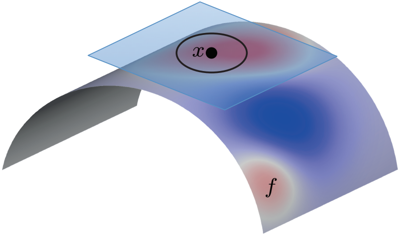

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- Positive semidefinite

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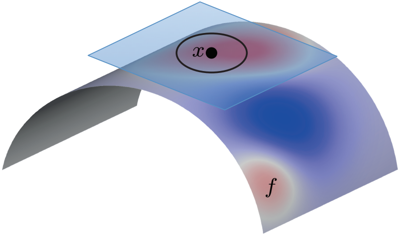

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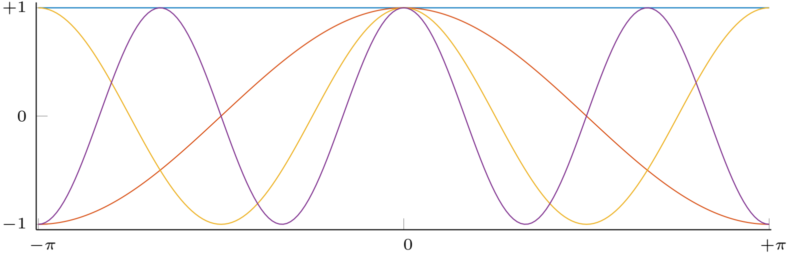
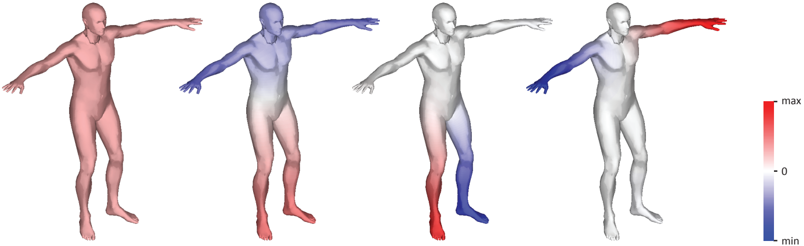
- Positive semidefinite $\Rightarrow$ non-negative eigenvalues

Laplacian eigenfunctions: Euclidean



First eigenfunctions of 1D Euclidean Laplacian = standard Fourier basis

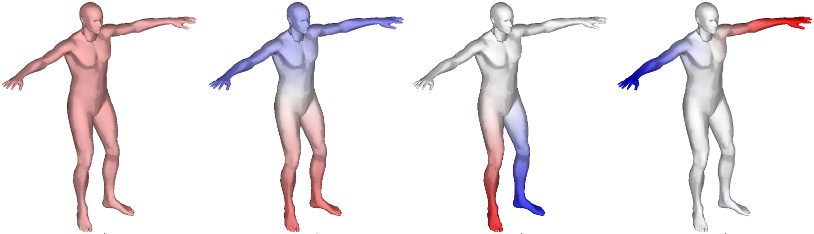
Laplacian eigenfunctions: non-Euclidean



ϕ_1 ϕ_2 ϕ_3 ϕ_4

First eigenfunctions of a manifold Laplacian = Fourier basis on manifolds

Laplacian eigenfunctions: non-Euclidean



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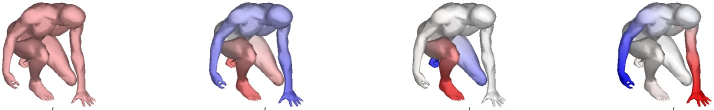
For shapes with simple spectrum, Laplacian eigenfunctions are invariant (up to sign) to isometric deformations, $\psi_i = \pm T\phi_i$



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Laplacian eigenbasis

- $\phi_1, \phi_2, \ldots$ is an **orthogonal basis** on $L^2(\mathcal{X})$, i.e. $\langle \phi_i, \phi_j \rangle_{L^2(\mathcal{X})} = \delta_{ij}$
- Smoothest orthogonal basis, due to minimization of the **Dirichlet energy**

$$\min_{\phi_i} \|\nabla \phi_i\|^2 \quad \text{s.t.} \quad \|\phi_i\| = 1, \quad i = 1, 2, \ldots$$
$$\phi_i \perp \text{span}\{\phi_1, \ldots, \phi_{i-1}\}$$

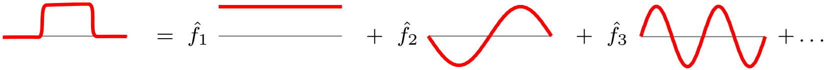
- Optimal basis for smooth signals
- **Intrinsic**, hence invariant under inelastic deformations (isometries)
- Non-Euclidean analogy of the **Fourier transform**

Aflalo et al. 2013, 2015

Fourier analysis (Euclidean spaces)

A (square-integrable) function $f : [-\pi, \pi] \to \mathbb{R}$ can be written as Fourier series

$$f(x) = \sum_{\omega} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi) e^{i\omega\xi} \mathrm{d}\xi \ e^{-i\omega x}$$



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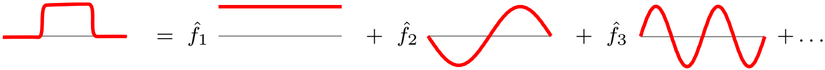
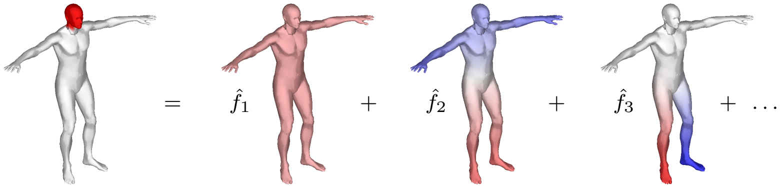


Fourier analysis (non-Euclidean spaces)

A (square-integrable) function $f : \mathcal{X} \to \mathbb{R}$ can be written as Fourier series

$$f(x) = \sum_{k \geq 1} \underbrace{\int_{\mathcal{X}} f(\xi)\phi_k(\xi)\mathrm{d}\xi}_{\hat{f}_k = \langle f, \phi_k \rangle_{L^2(\mathcal{X})}} \; \phi_k(x)$$



Heat diffusion on manifolds

Heat equation governs diffusion processes on manifolds:

$$\begin{cases} f_t(x,t) = -\Delta f(x,t) \\ f(x,0) = f_0(x) \qquad \text{initial condition} \end{cases}$$

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Solution of the heat equation expressed through the heat operator

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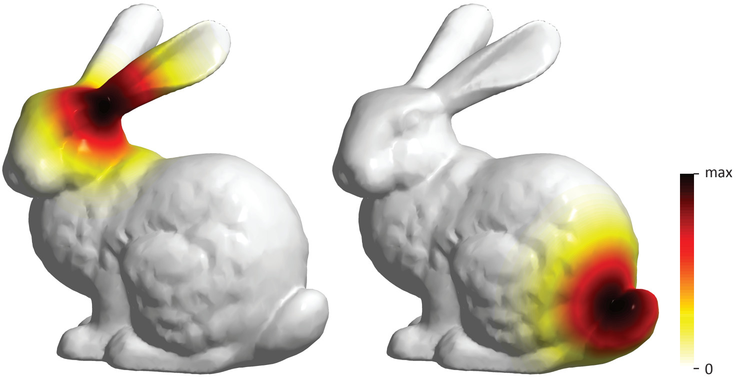
Heat kernel



Heat kernel $h_t(x, \cdot)$ at different points on a manifold

$\lambda =$ frequency

$\phi =$ Fourier atoms

λ = frequency

ϕ = Fourier atoms

What can be recovered from the Laplacian spectrum?

Heat trace expansion

$$\mathrm{trace}(e^{-t\Delta}) = \sum_{k\geq 1} e^{-t\lambda_k} = \sum_{k\geq 1} a_k t^k$$

for a 2-manifold (without boundary) allows to recover the following properties from the Laplacian spectrum:

Minakshisundaram, Pleijel 1949

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- Area: $a_0 = \mathrm{area}(\mathcal{X})$
- Total Gaussian curvature: $a_1 = \frac{1}{3}\int_{\mathcal{X}} K(x)\mathrm{d}x$

Minakshisundaram, Pleijel 1949

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for a 2-manifold (without boundary) allows to recover the following properties from the Laplacian spectrum:

- Area: $a_0 = \mathrm{area}(\mathcal{X})$
- Total Gaussian curvature: $a_1 = \frac{1}{3}\int_{\mathcal{X}} K(x)\mathrm{d}x$, and by Gauss-Bonnet theorem, the Euler characteristic: $a_1 = \frac{2\pi}{3}\chi(\mathcal{X})$

Minakshisundaram, Pleijel 1949

What can be recovered from the Laplacian spectrum?

Heat trace expansion

$$\mathrm{trace}(e^{-t\Delta}) = \sum_{k\geq 1} e^{-t\lambda_k} = \sum_{k\geq 1} a_k t^k$$

for a 2-manifold (without boundary) allows to recover the following properties from the Laplacian spectrum:

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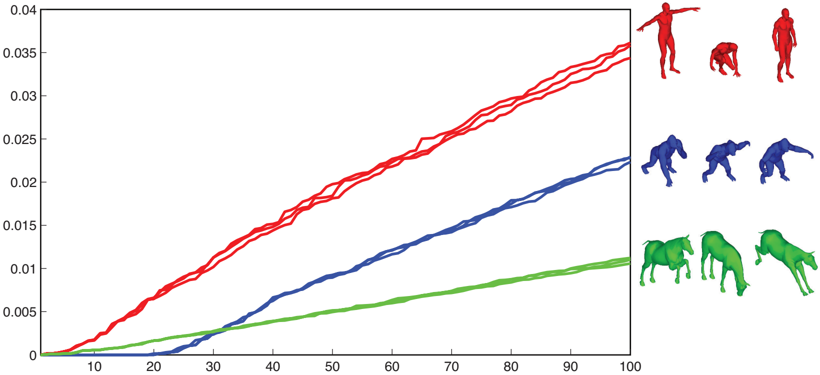
- ...

Minakshisundaram, Pleijel 1949

Shape DNA



Laplacian spectrum used as deformation-invariant shape descriptor

Reuter, Wolter, Peinecke 2005

isometric $\Longrightarrow$ isospectral

$$\textbf{isometric} \overset{?}{\Longleftrightarrow} \textbf{isospectral}$$

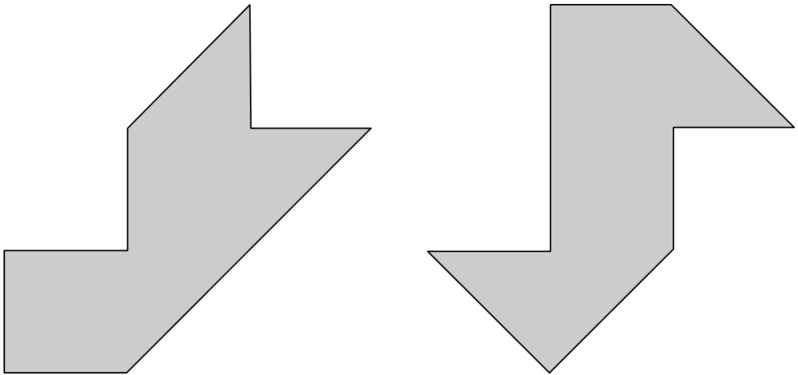
CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

> "La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

Counter-examples



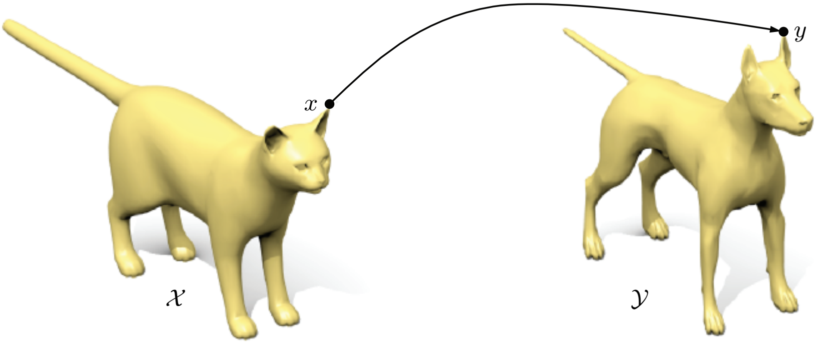
One cannot hear the shape of the drum!

Gordon, Web, Wolpert 1992

$\lambda =$ frequency

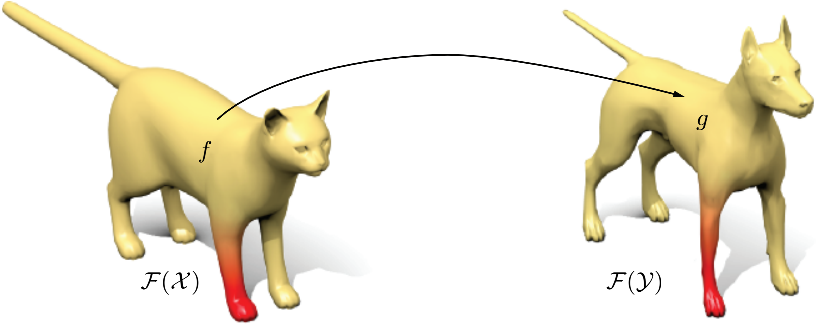
$\phi =$ Fourier atoms

Pointwise correspondence



Point-wise map $\tau\colon \mathcal{X} \to \mathcal{Y}$

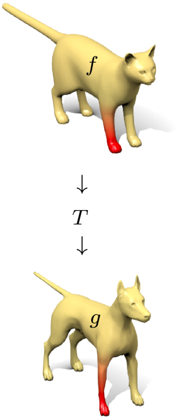
Functional correspondence



Functional map $T\colon \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{Y})$

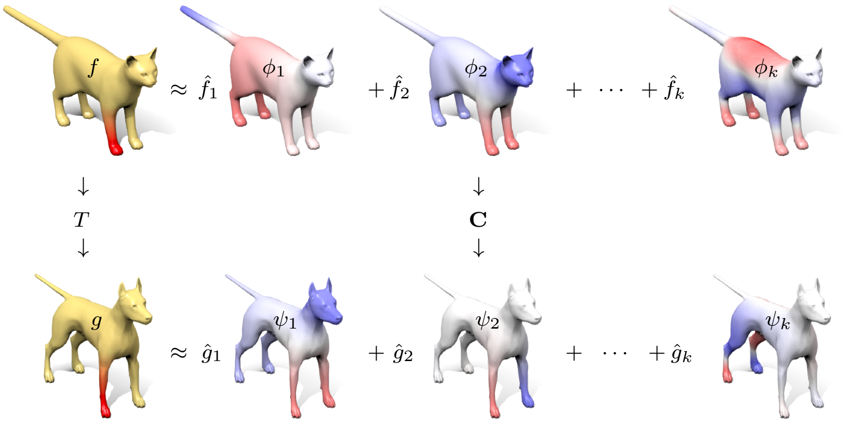
Ovsjanikov et al. 2012

Functional correspondence in spectral domain



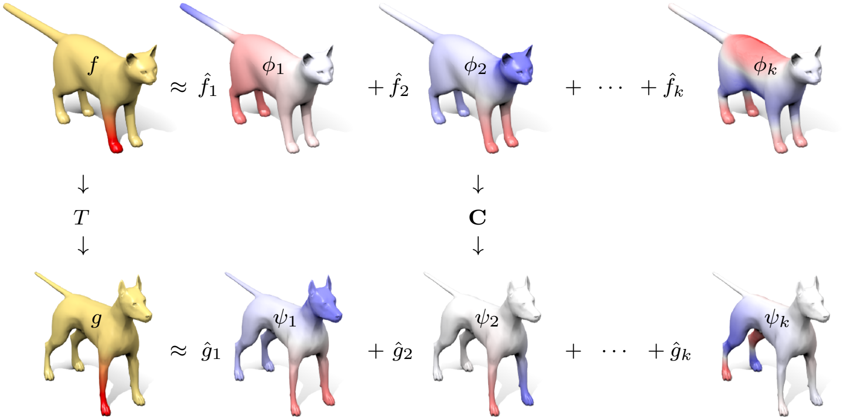
Ovsjanikov et al. 2012

Functional correspondence in spectral domain



Ovsjanikov et al. 2012

Functional correspondence in spectral domain

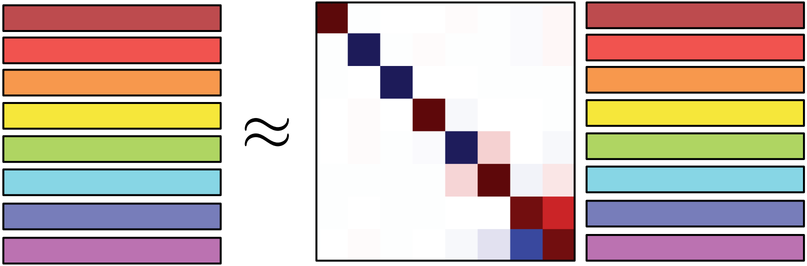


Functional correspondence boils down to a linear equation w.r.t. $\mathbf{C}$

$$g = Tf \qquad \Longleftrightarrow \qquad \hat{\mathbf{g}} = \mathbf{C}\hat{\mathbf{f}}$$

Ovsjanikov et al. 2012

Functional correspondence in spectral domain



$\hat{\mathbf{G}}_{k\times q} = \left(\langle \psi_i, g_j \rangle_{\mathcal{F}(\mathcal{Y})}\right)$ $\quad \mathbf{C}_{k\times k} \quad$ $\hat{\mathbf{F}}_{k\times q} = \left(\langle \phi_i, f_j \rangle_{\mathcal{F}(\mathcal{X})}\right)$

where $\hat{\mathbf{F}}$, $\hat{\mathbf{G}}$ are Fourier coefficients of corresponding 'probe' functions

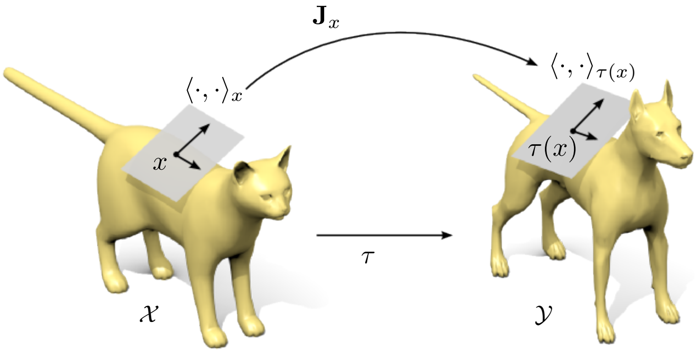
$$g_i \approx T f_i \qquad i = 1, \ldots, q \geq k$$

Ovsjanikov et al. 2012

Functional maps: Summary

- Intrinsic by construction
- Operate with Fourier coefficients, forget about specific discretization (can apply to meshes, point clouds, etc.)
- Advantage: replace intractable combinatorial problems with tractable linear algebra problems
- Disadvantage: hard to guarantee properties of maps (e.g. bijectivity)
- Some properties can be guaranteed: e.g. **area preservation** = **orthogonality** of the functional map ($\mathbf{C}^\top \mathbf{C} = \mathbf{I}$)

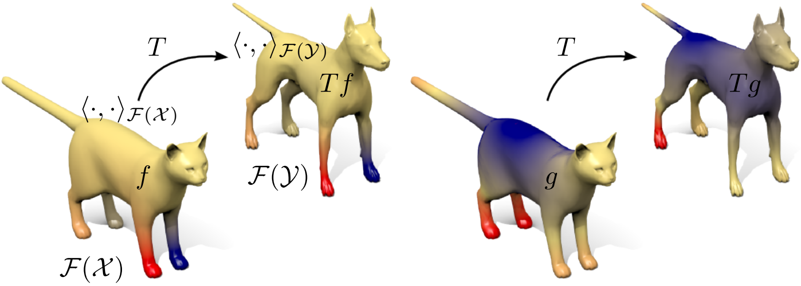
Shape difference operators



Distortions induced by a map = change of Riemannian metric
(inner product of **tangent vectors**)

Rustamov et al., 2013

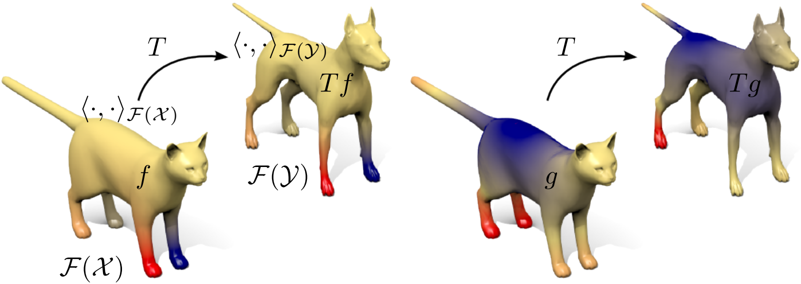
Shape difference operators



Distortions induced by a map = change of inner products of **functions**

Rustamov et al., 2013

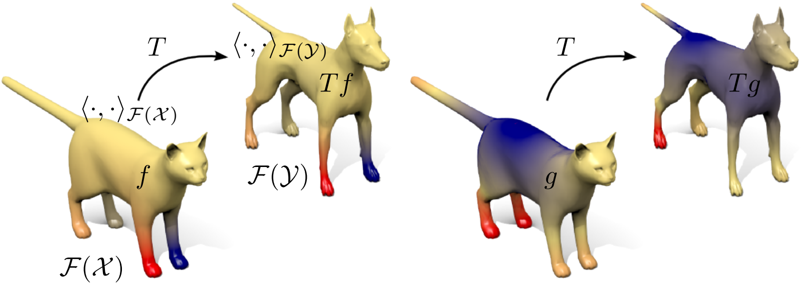
Shape difference operators



$$\langle f, g\rangle_{\mathcal{F}(\mathcal{X})} \neq \langle Tf, Tg\rangle_{\mathcal{F}(\mathcal{Y})}$$

Rustamov et al., 2013

Shape difference operators



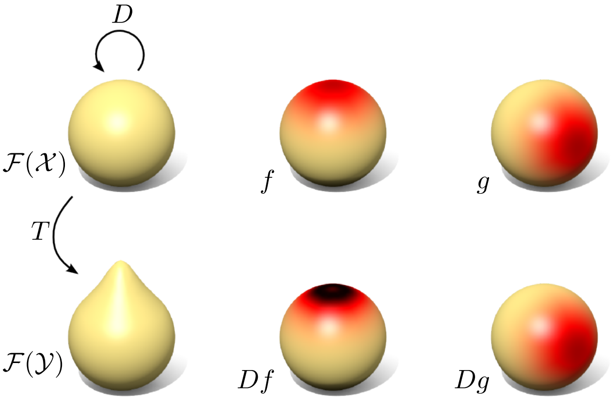
$$\langle f, g\rangle_{\mathcal{F}(\mathcal{X})} \neq \langle Tf, Tg\rangle_{\mathcal{F}(\mathcal{Y})}$$

Riesz theorem: there exists a unique self-adjoint linear operator $D\colon \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{X})$ such that

$$\langle Tf, Tg\rangle_{\mathcal{F}(\mathcal{Y})} = \langle f, Dg\rangle_{\mathcal{F}(\mathcal{X})}$$

Rustamov et al., 2013

Shape difference operators



- Captures the **difference** in the geometry of the two shapes
- **Depends** on choice of inner product

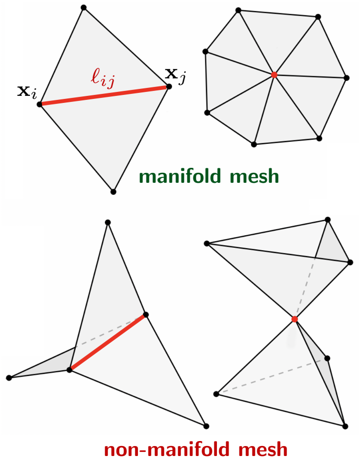
Rustamov et al., 2013

Discretization

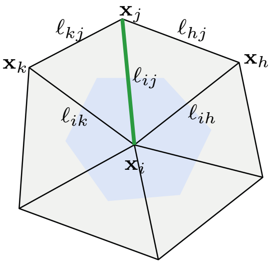
Manifold meshes

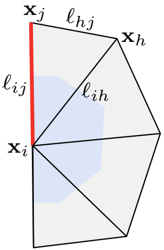
Surface discretized as a embedded triangular mesh $(\mathbf{X}, \mathcal{E}, \mathcal{F})$

- **Nodes:** points in 3D ($n \times 3$ matrix $\mathbf{X}$ of coordinates)
- **Edges:** (i, j), weighted by metric $\ell_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|$
- **Faces:** (i, j, k) s.t. $(i, j), (i, k), (k, j) \in \mathcal{E}$
- Manifoldness assumption:
  - Each edge is shared by two triangles
  - Boundary of triangles incident on each node forms a single loop of edges



Laplacian discretization





Cotangent Laplacian $\mathbf{\Delta} = \mathbf{A}^{-1}\mathbf{W}$ expressed in terms of discrete metric $\ell_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|$ where

$$w_{ij} = \begin{cases} \frac{-\ell_{ij}^2+\ell_{jk}^2+\ell_{ki}^2}{8A_{ijk}} + \frac{-\ell_{ij}^2+\ell_{jh}^2+\ell_{hi}^2}{8A_{ijh}} & \text{if } e_{ij} \in \mathcal{E}_\text{i} \\ \frac{-\ell_{ij}^2+\ell_{jh}^2+\ell_{hi}^2}{8A_{ijh}} & \text{if } e_{ij} \in \mathcal{E}_\text{b} \\ -\sum_{k \neq i} w_{ik} & \text{if } i = j \end{cases} \qquad \mathbf{A} = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

where A_{ijk} is area of triangle ijk and $a_i = \frac{1}{3} \sum_{ijk: ij, ik \in \mathcal{E}} A_{ijk}$

Meyer et al. 2003

Shape difference discretization

- **area**-based

$$\langle f, g\rangle_{L^2(\mathcal{X})} = \int_{\mathcal{X}} f(x)g(x)\mathrm{d}x$$

$$\mathbf{D} = \mathbf{V}_{\mathcal{X},\mathcal{Y}} = \mathbf{A}_{\mathcal{X}}^{-1}\mathbf{T}^\top\mathbf{A}_{\mathcal{Y}}\mathbf{T}$$

Rustamov et al., 2013

Shape difference discretization

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- **conformal**-based

$$\langle f, g\rangle_{H^1(\mathcal{X})} = \int_{\mathcal{X}} \langle \nabla f(x), \nabla g(x)\rangle_x \mathrm{d}x$$

$$\mathbf{D} = \mathbf{R}_{\mathcal{X},\mathcal{Y}} = \mathbf{W}_{\mathcal{X}}^{\dagger}\mathbf{T}^\top \mathbf{W}_{\mathcal{Y}}\mathbf{T}$$

Rustamov et al., 2013

Shape difference discretization

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$$\langle f, g \rangle_{L^2(\mathcal{X})} = \int_{\mathcal{X}} f(x) g(x) \mathrm{d}x$$

$$\mathbf{D} = \mathbf{V}_{\mathcal{X},\mathcal{Y}} = \mathbf{A}_{\mathcal{X}}^{-1} \mathbf{T}^\top \mathbf{A}_{\mathcal{Y}} \mathbf{T}$$

- $\mathbf{V} = \mathbf{I}$: area-preserving maps

- **conformal**-based

$$\langle f, g \rangle_{H^1(\mathcal{X})} = \int_{\mathcal{X}} \langle \nabla f(x), \nabla g(x) \rangle_x \mathrm{d}x$$

$$\mathbf{D} = \mathbf{R}_{\mathcal{X},\mathcal{Y}} = \mathbf{W}_{\mathcal{X}}^{\dagger} \mathbf{T}^\top \mathbf{W}_{\mathcal{Y}} \mathbf{T}$$

- $\mathbf{R} = \mathbf{I}$: angle-preserving (conformal) map

- $\mathbf{V} = \mathbf{R} = \mathbf{I}$: isometric map

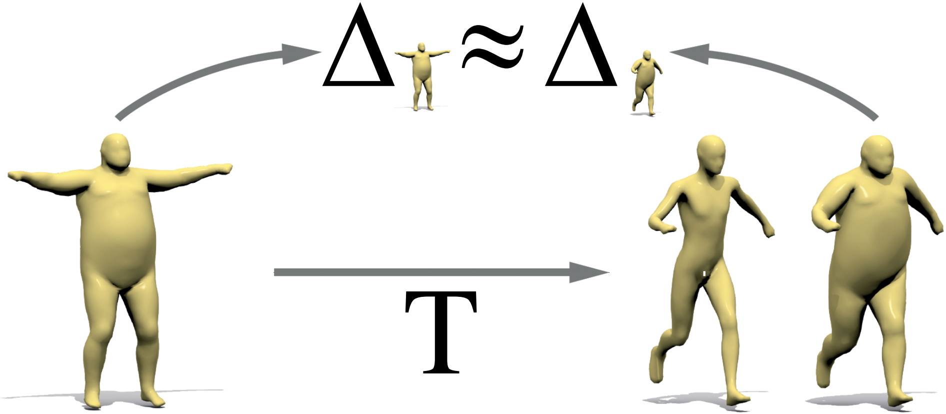
Rustamov et al., 2013

EUROGRAPHICS 2015 / O. Sorkine-Hornung and M. Wimmer
(Guest Editors)

Volume 34 (2015), Number 2

Shape-from-Operator: Recovering Shapes from Intrinsic Operators

Davide Boscaini, Davide Eynard, Drosos Kourounis, and Michael M. Bronstein
Università della Svizzera Italiana (USI), Lugano, Switzerland



Shape-from-Operator problems

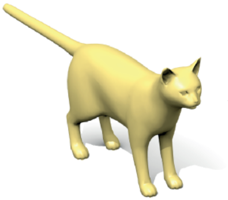
$$\longleftarrow \mathbf{O}_0$$

Generic **Shape-from-Operator (SfO)** problem: given some intrinsic operator $\mathbf{O}_0$, find an embedding $\mathbf{X}$ by minimizing some cost function

$$\min_{\mathbf{X}\in\mathbb{R}^{n\times 3}} E\big(\mathbf{O}(\ell(\mathbf{X})), \mathbf{O}_0\big)$$

Boscaini et B 2014

Shape-from-Operator problems



$$\longleftarrow \mathbf{O}_0$$

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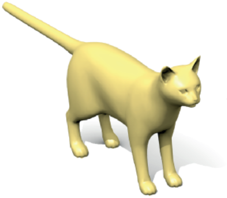
$$\min_{\mathbf{X}\in\mathbb{R}^{n\times 3}} E\big(\mathbf{O}(\ell(\mathbf{X})), \mathbf{O}_0\big)$$

$\mathbf{O}$ depends on $\mathbf{X}$ indirectly through the discrete metric $\ell(\mathbf{X})$, very hard for optimization!

Boscaini et B 2014

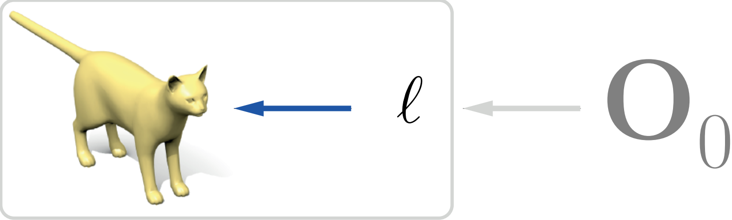
Shape-from-Operator problems

 $\longleftarrow \quad \ell \quad \longleftarrow \quad \mathbf{O}_0$

- **Metric-from-Operator (MfO):** $\min_{\ell} E\big(\mathbf{O}(\ell), \mathbf{O}_0\big)$ s.t. Δ inequality
- **Shape-from-Metric (SfM):** $\min_{\mathbf{X} \in \mathbb{R}^{n \times 3}} \sum_{(i,j) \in \mathcal{E}} (\|\mathbf{x}_i - \mathbf{x}_j\| - \ell_{ij})^2,$

Boscaini et B 2014

Shape-from-Operator problems



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Boscaini et B 2014

Shape-from-metric

Special setting of **MDS**: given a metric ℓ, find its Euclidean realization by minimizing the stress

$$\min_{\mathbf{X}\in\mathbb{R}^{n\times 3}} \sum_{i,j=1}^{n} v_{ij}(\|\mathbf{x}_i - \mathbf{x}_j\| - \ell_{ij})^2,$$

where

$$v_{ij} = \begin{cases} 1 & \text{if } (i,j) \in \mathcal{E}, \\ 0 & \text{otherwise} \end{cases}$$

Leeuw et al., 1977

Shape-from-metric

Special setting of **MDS**: given a metric ℓ, find its Euclidean realization by minimizing the stress

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where

$$v_{ij} = \begin{cases} 1 & \text{if } (i,j)\in\mathcal{E}, \\ 0 & \text{otherwise} \end{cases}$$

SMACOF algorithm: fixed point iteration of the form

$$\mathbf{X} \leftarrow \mathbf{Z}^{\dagger}\mathbf{B}(\mathbf{X})\mathbf{X}$$

where

$$\mathbf{Z} = \begin{cases} -v_{ij} & \text{if } i \neq j, \\ \sum_{i\neq j} v_{ij} & \text{if } i = j \end{cases} \qquad \mathbf{B}(\mathbf{X}) = \begin{cases} -\frac{v_{ij}\ell_{ij}}{\|\mathbf{x}_i - \mathbf{x}_j\|} & \text{if } i\neq j \text{ and } \mathbf{x}_i \neq \mathbf{x}_j, \\ 0 & \text{if } i\neq j \text{ and } \mathbf{x}_i = \mathbf{x}_j, \\ \sum_{i\neq j} b_{ij} & \text{if } i = j \end{cases}$$

Leeuw et al., 1977

From operators to shapes

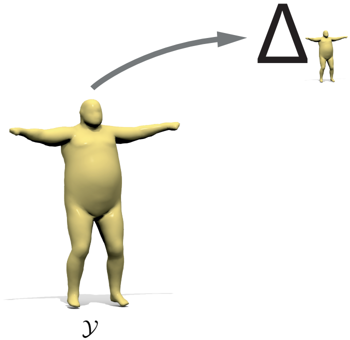
$$\ell \longleftarrow \mathbf{O}_0$$

- **Metric-from-Operator (MfO):** $\min_{\ell} E\big(\mathbf{O}(\ell), \mathbf{O}_0\big)$ s.t. Δ inequality

- **Shape-from-Metric (SfM):** $\min_{\mathbf{X} \in \mathbb{R}^{n \times 3}} \sum_{(i,j) \in \mathcal{E}} (\|\mathbf{x}_i - \mathbf{x}_j\| - \ell_{ij})^2,$

Boscaini et B 2014

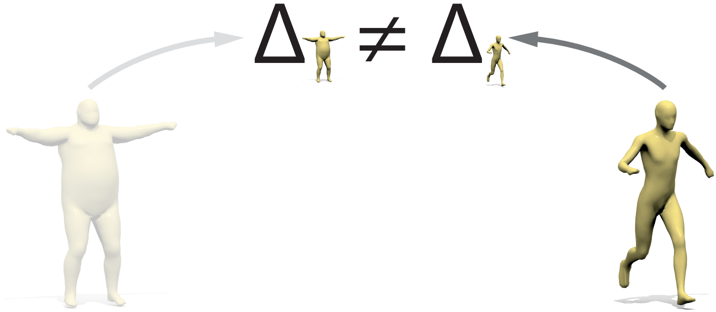
Shape-from-Laplacian



Given a reference Laplacian operator $\mathbf{\Delta}_{\mathcal{Y}}$

Boscaini et B 2014

Shape-from-Laplacian

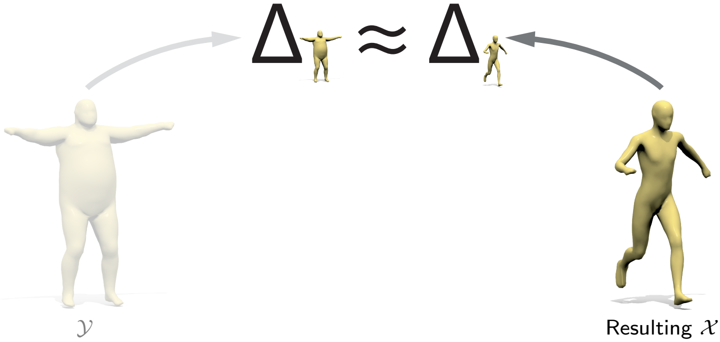


Given a reference Laplacian operator $\mathbf{\Delta}_{\mathcal{Y}}$, and a corresponding initial shape $\mathcal{X}$, deform $\mathcal{X}$ by minimizing

$$\min_{\mathbf{X}\in\mathbb{R}^{n\times 3}} \|\mathbf{A}^{-1}(\ell(\mathbf{X}))\mathbf{W}(\ell(\mathbf{X})) - \mathbf{\Delta}_{\mathcal{Y}}\|$$

Boscaini et B 2014

Shape-from-Laplacian



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$$\min_{\mathbf{X}\in\mathbb{R}^{n\times 3}} \|\mathbf{A}^{-1}(\ell(\mathbf{X}))\mathbf{W}(\ell(\mathbf{X})) - \mathbf{\Delta}_{\mathcal{Y}}\|$$

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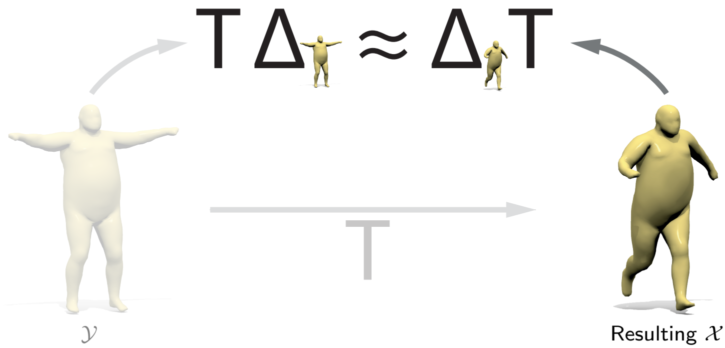
Shape-from-Laplacian



Given a reference Laplacian operator $\mathbf{\Delta}_{\mathcal{Y}}$, and a initial shape $\mathcal{X}$ related by functional correspondence $\mathbf{T}$, deform $\mathcal{X}$ by minimizing

$$\min_{\mathbf{X}\in\mathbb{R}^{n\times 3}} \|\mathbf{T}\mathbf{A}^{-1}(\ell(\mathbf{X}))\mathbf{W}(\ell(\mathbf{X})) - \mathbf{\Delta}_{\mathcal{Y}}\mathbf{T}\|$$

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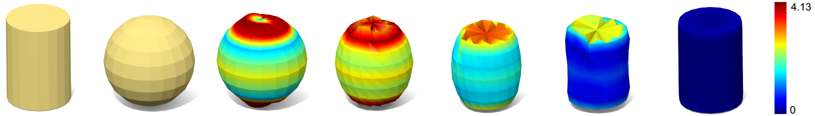
Shape-from-Laplacian convergence

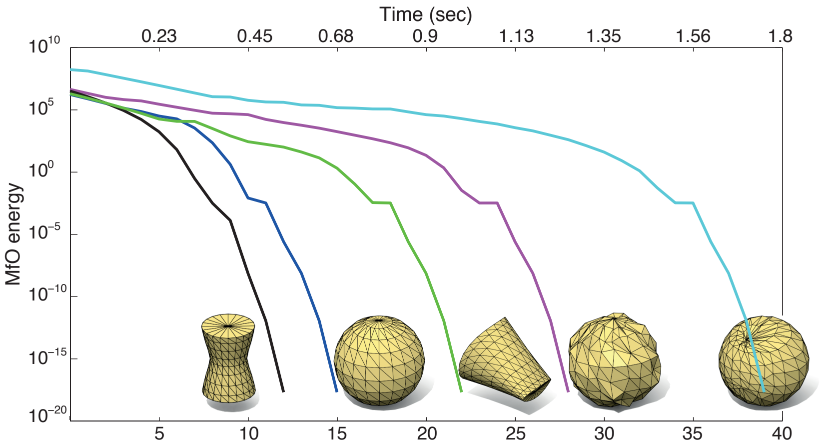


Reference Y Initialization X Iteration 1 4 6 7 9

Convergence of our method in the shape-from-Laplacian optimization problem.
Colors show vertex-wise MfO energy contribution

Boscaini et B 2014

Shape-from-Laplacian convergence



Convergence of our method in the shape-from-Laplacian optimization problem using different initializations.

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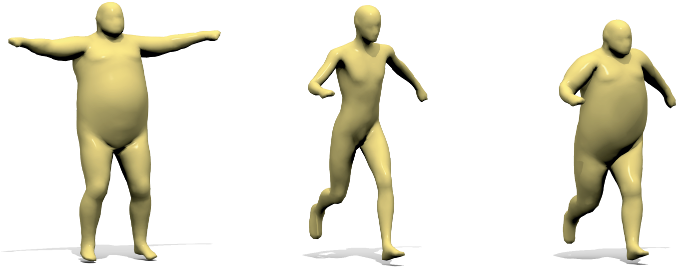
Style transfer by shape-from-Laplacian

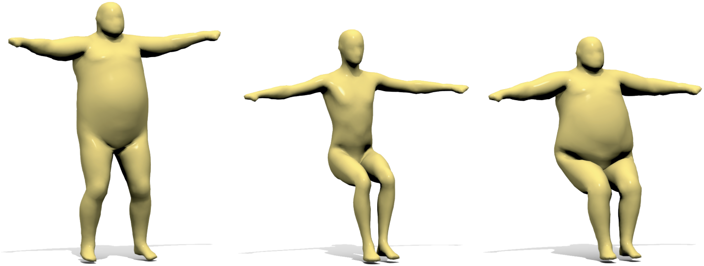


Reference Initialization Result

"Modify $\mathcal{X}$ such that $\mathbf{\Delta}_{\mathcal{X}}$ becomes as similar as possible to reference Laplacian $\mathbf{\Delta}_{\mathcal{Y}}$"

Boscaini et B 2014

Style transfer by shape-from-Laplacian



Reference	Initialization	Result

"Modify $\mathcal{X}$ such that $\boldsymbol{\Delta}_{\mathcal{X}}$ becomes as similar as possible to reference Laplacian $\boldsymbol{\Delta}_{\mathcal{Y}}$"

Boscaini et B 2014

Style transfer by shape-from-Laplacian

Reference Initialization Result

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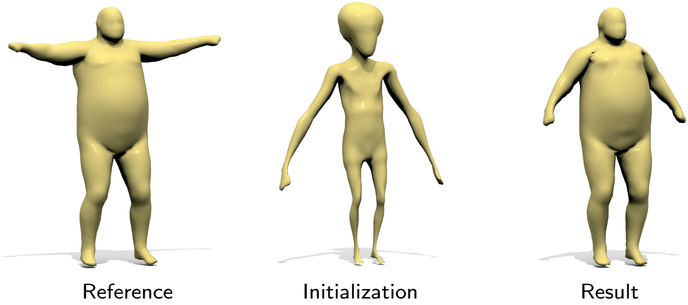
Boscaini et B 2014

Style transfer by shape-from-Laplacian



"Modify $\mathcal{X}$ such that $\mathbf{\Delta}_{\mathcal{X}}$ becomes as similar as possible to reference Laplacian $\mathbf{\Delta}_{\mathcal{Y}}$"

Boscaini et B 2014

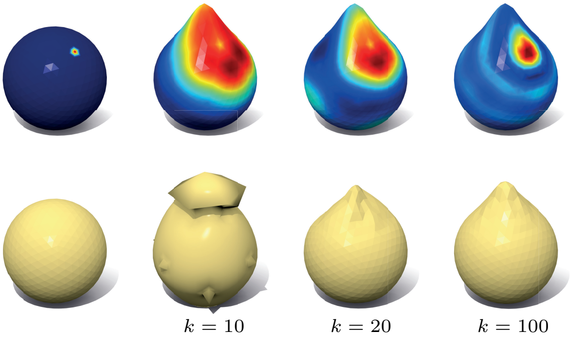
Sensitivity to map quality

Functional map approximated as a matrix $\mathbf{T} \approx \mathbf{\Psi}_k \mathbf{C}^\top \mathbf{\Phi}_k^\top$ of rank k using the first functions in Fourier expansion (larger k = better map)

Ovsjanikov et al. 2012

Sensitivity to map quality

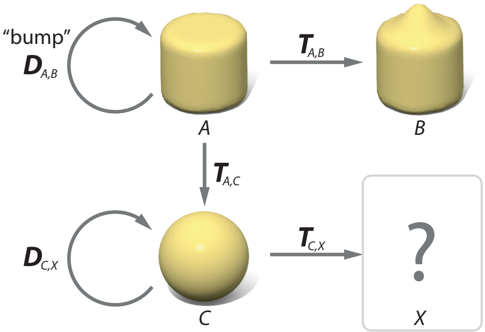
Functional map approximated as a matrix $\mathbf{T} \approx \mathbf{\Psi}_k \mathbf{C}^\top \mathbf{\Phi}_k^\top$ of rank k using the first functions in Fourier expansion (larger k = better map)



Shape-from-Laplacian result for different quality of the map $\mathbf{T}$ (initial shape: sphere, reference Laplacian: bumped sphere)

Ovsjanikov et al. 2012; Boscaini et B 2014

Shape-from-difference operator

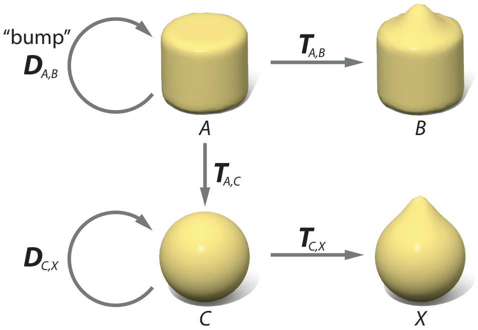


Deform initial shape $\mathcal{X}$ to make it different from $\mathcal{C}$ same way as $\mathcal{B}$ is different from $\mathcal{A}$

$$\min_{\mathbf{X}\in\mathbb{R}^{n\times 3}} \quad \|\mathbf{D}_{\mathcal{C},\mathcal{X}}(\ell(\mathbf{X}))\mathbf{T}_{\mathcal{A},\mathcal{C}} - \mathbf{T}_{\mathcal{A},\mathcal{C}}\mathbf{D}_{\mathcal{A},\mathcal{B}}\|$$

Boscaini et B 2014

Shape-from-difference operator



Deform initial shape $\mathcal{X}$ to make it different from $\mathcal{C}$ same way as $\mathcal{B}$ is different from $\mathcal{A}$

$$\min_{\mathbf{X}\in\mathbb{R}^{n\times 3}} \quad \|\mathbf{D}_{\mathcal{C},\mathcal{X}}(\ell(\mathbf{X}))\mathbf{T}_{\mathcal{A},\mathcal{C}} - \mathbf{T}_{\mathcal{A},\mathcal{C}}\mathbf{D}_{\mathcal{A},\mathcal{B}}\|$$

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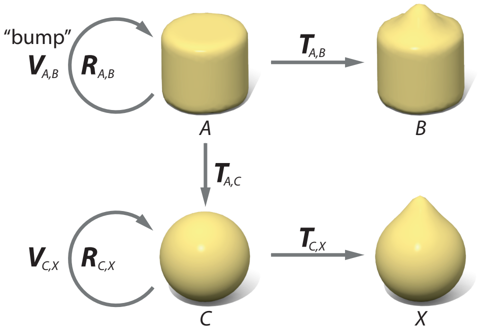
Shape-from-difference operator



Deform initial shape $\mathcal{X}$ to make it different from $\mathcal{C}$ same way as $\mathcal{B}$ is different from $\mathcal{A}$

$$\min_{\mathbf{X}\in\mathbb{R}^{n\times 3}} \quad \mu\|\mathbf{A}_{\mathcal{C}}^{-1}\mathbf{T}_{\mathcal{C},\mathcal{X}}^{\top}\mathbf{A}(\ell(\mathbf{X}))\mathbf{T}_{\mathcal{C},\mathcal{X}}\mathbf{T}_{\mathcal{A},\mathcal{C}} - \mathbf{T}_{\mathcal{A},\mathcal{C}}\mathbf{V}_{\mathcal{A},\mathcal{B}}\| + (1-\mu)\|\mathbf{W}_{\mathcal{C}}^{\dagger}\mathbf{T}_{\mathcal{C},\mathcal{X}}^{\top}\mathbf{W}(\ell(\mathbf{X}))\mathbf{T}_{\mathcal{C},\mathcal{X}}\mathbf{T}_{\mathcal{A},\mathcal{C}} - \mathbf{T}_{\mathcal{A},\mathcal{C}}\mathbf{R}_{\mathcal{A},\mathcal{B}}\|$$

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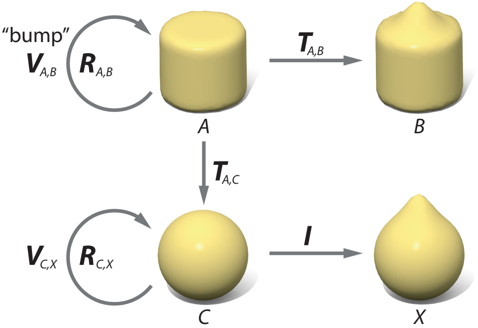
Shape-from-difference operator



Deform initial shape $\mathcal{X}$ to make it different from $\mathcal{C}$ same way as $\mathcal{B}$ is different from $\mathcal{A}$

$$\min_{\mathbf{X}} \quad \mu \|\mathbf{A}_C^{-1}\mathbf{A}(\ell(\mathbf{X}))\mathbf{T}_{A,C} - \mathbf{T}_{A,C}\mathbf{V}_{A,B}\| + (1-\mu)\|\mathbf{W}_C^{\dagger}\mathbf{W}(\ell(\mathbf{X}))\mathbf{T}_{A,C} - \mathbf{T}_{A,C}\mathbf{R}_{A,B}\|$$

(initializing $\mathcal{X} = \mathcal{C}$ we have $\mathbf{T}_{\mathcal{C},\mathcal{X}} = \mathbf{I}$)

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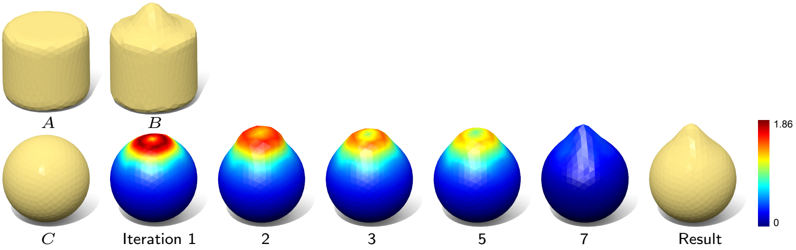
Shape-from-difference convergence



Convergence of our method in the shape-from-difference optimization problem
Colors show vertex-wise MfO energy contribution

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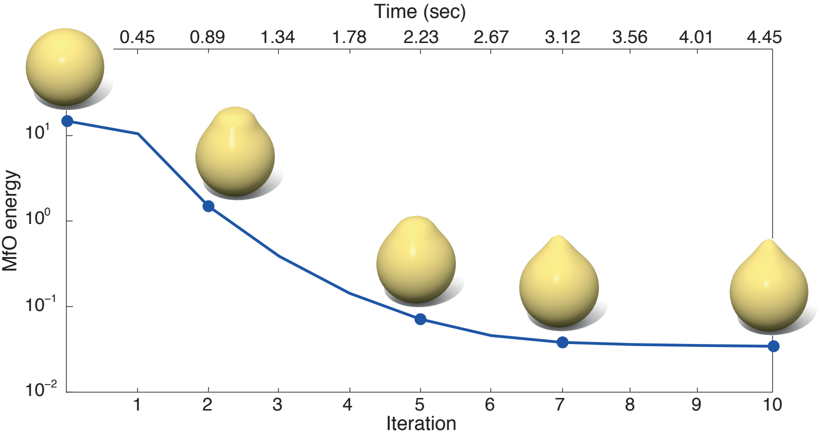
Shape-from-difference convergence



Convergence of our method in the shape-from-difference optimization problem.

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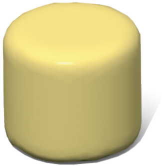
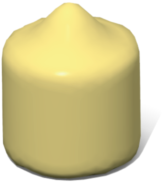
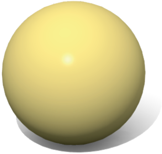
Analogy synthesis by shape-from-difference



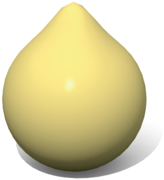
$\mathcal{A}$



$\mathcal{B}$



$\mathcal{C}$



$\mathcal{X} = \mathcal{C} + (\mathcal{B} - \mathcal{A})$

"Find $\mathcal{X}$ such that the difference operator between $\mathcal{C}, \mathcal{X}$ is as similar as possible to the given difference operator between $\mathcal{A}, \mathcal{B}$"

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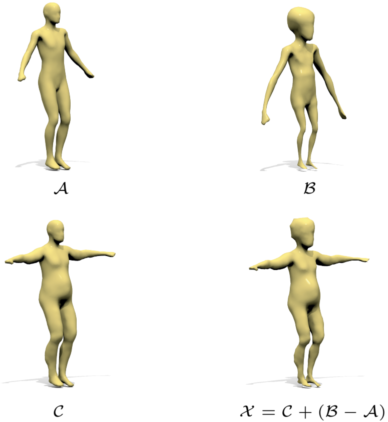
Analogy synthesis by shape-from-difference



"Find $\mathcal{X}$ such that the difference operator between $\mathcal{C}, \mathcal{X}$ is as similar as possible to the given difference operator between $\mathcal{A}, \mathcal{B}$"

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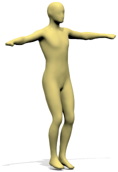
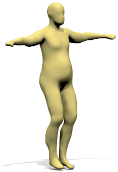
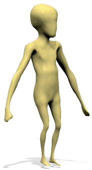
Analogy synthesis by shape-from-difference



$\mathcal{A}$



$\mathcal{B}$



$\mathcal{C}$

$\mathcal{X} = \mathcal{C} + (\mathcal{B} - \mathcal{A})$

"Find $\mathcal{X}$ such that the difference operator between $\mathcal{C}, \mathcal{X}$ is as similar as possible to the given difference operator between $\mathcal{A}, \mathcal{B}$"

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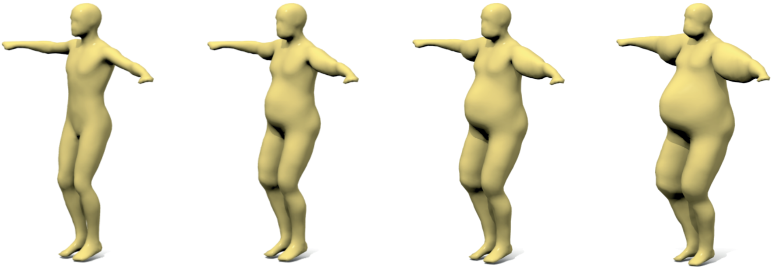
Shape exaggeration



$\mathcal{A}$ $\quad$ $\mathcal{B}$ $\quad$ '$\mathcal{B} + (\mathcal{B} - \mathcal{A})$' $\quad$ '$\mathcal{B} + 2(\mathcal{B} - \mathcal{A})$'

Shape exaggeration obtained by applying the difference operator between $\mathcal{A}, \mathcal{B}$ to $\mathcal{B}$ several times

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Summary

- Laplacian eigenvectors form an orthogonal basis for functions on a manifold
- Lifting to space of functions on manifolds: replace combinatorial problems with linear algebra problems
- Operate with Fourier coefficients, forget about specific discretization
- Represent shapes as operators (contains full information up to some class of transformations, e.g. isometries or conformal maps)
- Recover shapes from operators
- Interesting links to Geometric Deep Learning