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Supplementary material: Curves with different turning numbers

We fill in the details of the discussion of morphing curves with different turning numbers from the main text. We assume $\tau(\mathbf{p}^1) > \tau(\mathbf{p}^0) > 0$ and denote $k = \tau(\mathbf{p}^1) - \tau(\mathbf{p}^0)$. We explain how we choose the vertex at which flips occur when $k = 1$, and then how this is generalized to the case $k > 1$. We will conclude with a discussion of the issue of feasibility.

Let us consider the case $k = 1$. and assume that we use our morphing algorithm without any modifications. We now show pinching always occurs at the first vertex only. When well defined, the external angles $\theta_j^t,\ j = 1, \dots n-1$ satisfy (define $\varphi_0 \equiv \varphi_{n-1}$)

$$\varphi_j^t - \varphi_{j-1}^t = \theta_j^t - 2\pi n_j^t, \quad \theta_j^t \in (-\pi, \pi),\ n_j^t \in \mathbb{Z}$$

For $j > 1$, θ_j^t is the linear interpolation of θ_j^0 and θ_j^1, and $n_j^t \equiv 0$. When $\tau(\mathbf{p}^0) = \tau(\mathbf{p}^1)$, θ_1^t is again the linear interpolation of θ_j^0 and θ_j^1, and $n_j^t \equiv \tau(\mathbf{p}^0) = \tau(\mathbf{p}^1)$. However, in our case there is some t_1 at which n_j^t changes from $\tau(\mathbf{p}^0)$ to $\tau(\mathbf{p}^1)$. The external angle θ_1^t linearly decreases from θ_1^0 to $-\pi$ on the interval $(0, t_1)$, and then from π to θ_1^1 on the interval $(t_1, 1)$. We conclude that the unmodified morph has a single pinching point at $(t_1, 1)$.

We choose the starting point of the curve so as to minimize work in angular space. We note that the total angular change at $\mathbf{p}_1$, computed by summing the change on $(0, t_1)$ and $(t_1, 1)$, is given by

$$\Delta(\theta_1) = |\theta_1^0 - (-\pi)| + |\pi - \theta_1^1| = 2\pi + \theta_1^0 - \theta_1^1$$

While the total angular change at $\mathbf{p}_j,\ j = 2, \dots n-1$ is given by $\Delta(\theta_j) = |\theta_j^1 - \theta_j^0|$. Thus to minimize the total angular change of all vertices

$$\sum_{j=1}^{n-1} \Delta(\theta_j) = 2\pi + \left(\theta_1^0 - \theta_1^1\right) + \sum_{j=2}^{n-1} |\theta_j^1 - \theta_j^0|$$

the optimal choice for a first vertex is

$$j^* = \text{argmin}_{j=1,\dots,n-1} \theta_j^0 - \theta_j^1$$

The successfulness of this approach is illustrated in Figure 12.

When $k > 1$, directly applying our interpolation scheme causes pinching at the first vertex at k different times, and the total angular change is

$$\Delta(\theta^1) = 2\pi k + \theta_1^1 - \theta_1^0 > 2\pi(k-1)$$

It therefore seems more reasonable to disperse the pinching along k different vertices. This can be done by choosing a subset $J \subseteq \{1, 2, \dots, n-1\}$ with $|J| = k$ and defining

$$\tilde{\theta}_j^1 = \theta_j^1$$

$$\tilde{\theta}_j^0 = \begin{cases} \theta_j^0 + 2\pi & \text{if } j \in J \\ \theta_j^0 & \text{if } j \notin J \end{cases}$$

$$\tilde{\theta}_j^t = (1-t)\tilde{\theta}_j^0 + t\tilde{\theta}_j^1$$

Note that for every t, $\sum_j \tilde{\theta}_j^t = 2\pi\tau(\mathbf{c}^1)$. Therefore

$$\theta_j^t = \begin{cases} \tilde{\theta}_j^t & \text{if } \tilde{\theta}_j^t < \pi \\ \tilde{\theta}_j^t - 2\pi & \text{if } \tilde{\theta}_j^t \geq \pi \end{cases} \tag{0.1}$$

coincide with θ_j^0, θ_j^1 when $t = 0, 1$ and satisfy

$$\sum \theta_j^t \in 2\pi\mathbb{Z} \tag{0.2}$$

Now choose φ_1^t by linearly interpolating φ_1^0, φ_1^1, and then recursively reconstruct φ_j^t via $\varphi_{j+1}^t = \varphi_j^t + \theta_j^t$, and find the lengths L_j^t via (5.1).

(0.2) implies that the external angles of the constructed curves will indeed be θ_j^t as our notation suggests. The total angular change at a vertex $\mathbf{p}_j$ with $j \notin J$ will be $\Delta(\theta_j) = |\theta_j^1 - \theta_j^0|$. If $j \in J$, then for $t_j \in (0, 1)$ satisfying $\tilde{\theta}_j^{t_j} = \pi$, (t_j, j) will be a pinching point, and the total angular change will be

$$\Delta(\theta_j) = 2\pi + \theta_j^0 - \theta_j^1$$

We conclude that to minimize $\sum_{j=1}^{n-1} \Delta(\theta_j)$ the optimal choice of J will be the k indices for which the value of $\theta_j^0 - \theta_j^1$ is minimal.

We note that when $k = 1$ this algorithm coincides with the strategy described in the beginning of this text.

Feasibility. We now discuss the feasibility of (5.1) for $t \in [0,1]$ when φ_j^t are constructed as described here. For $t \neq t_j$ feasibility holds just as in the regular case. At each t_j there necessarily must be consecutive angles $\theta_p, \theta_{p+1}, \dots, \theta_q$ (this may be a cyclic consecutive sequence of angles $\theta_p, \theta_{p+1}, \dots, \theta_{n-1}, \theta_1, \dots, \theta_q$)with $|\theta_l| < \pi$ who sum to more than π, since $\sum_{j=1}^{n-1} \theta_j \geq \tau(\mathbf{c}_0) \geq 2\pi$. By Lemma 5.3 $e^{i\varphi_p}, e^{i\varphi_{p+1}}, \dots, e^{i\varphi_q}$ span $\mathbb{R}^2$ and thus a positive solutions $L_j^{t_j} > 0$ for $\sum L_j^{t_j} e^{i\varphi_j^{t_j}} = 0$ can be constructed as in the proof of Theorem 4.1.