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Mirela Ben-Chen and Ligang Liu
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Supplementary material: Proofs

Approximation by piecewise linear curves. We prove our claim from Section 4 that the approximations of closed regular curves $\mathbf{c}$ by piecewise linear curves $\mathbf{c}_n$ converges, in the sense that for large enough n, $\mathbf{c}_n$ is regular and has the same turning number as $\mathbf{c}$. $\mathbf{c}_n$ are obtained by sampling $\mathbf{c}$ at points $0 = \mathfrak{s}_0 < \mathfrak{s}_1 < \ldots < \mathfrak{s}_n = 1$ which include the singular points of $\mathbf{c}$, and such that

$$\max_{0 \leq j \leq n-1} \mathfrak{s}_{j+1} - \mathfrak{s}_j \xrightarrow{n} 0$$

$\dot{\mathbf{c}}$ and $\dot{\mathbf{c}}_n$ are piecewise continuous $L^\infty([0,1])$ functions, and it is not difficult to show that $\dot{\mathbf{c}}_n \xrightarrow{L^\infty} \dot{\mathbf{c}}$.

Fix some $\varepsilon > 0$. There is a finite open cover of $[0,1]$ $U_1, \ldots U_k$ and vectors $\xi_1, \ldots \xi_k \in S^1$ such that

$$\langle \dot{\mathbf{c}}(s), \xi_j \rangle \geq \varepsilon, \text{ for almost every } s \in U_j$$

Thus, for all large enough n,

$$\langle \dot{\mathbf{c}}_n(s), \xi_j \rangle > 0, \text{ for almost every } s \in U_j \tag{0.1}$$

It follows that $\mathbf{c}_n$ are regular. They are also closed by construction. Finally, linearly interpolate $\dot{\mathbf{c}}_n$ and $\dot{\mathbf{c}}$ and denote the result by $\dot{\mathbf{c}}_n^t$. $\dot{\mathbf{c}}_n^t$ satisfies the convex conditions (0.1) and $\int \dot{\mathbf{c}}_n^t = 0$. Integrating $\dot{\mathbf{c}}_n^t$ we get a regular homotopy $\mathbf{c}_n^t$ between $\mathbf{c}_n$ and $\mathbf{c}$, which implies that both curves have the same turning number.

Proof of Lemma 6.1 For a proof that a convex curve with $\tau(\mathbf{c}) = 1$ must have a positive curvature function κ and that a simple closed curve with $\tau(\mathbf{c}) = 1$ and $\kappa \geq 0$ is convex, see [Csi].

Since $\kappa = \frac{\dot{\varphi}}{r}$, $\kappa \geq 0$ is equivalent to φ being non-decreasing. It thus remains only to show that a closed regular curve $\mathbf{c}$ with $\tau(\mathbf{c}) = 1$ and φ non-decreasing is convex. This too seems to be known, but we were not able to find a full reference so we prove this here.

It is sufficient to show that if $\mathbf{c}(0) = (x(0), y(0)) = (0,0)$ and $\varphi(0) = 0$ then for $0 < s < 1$, $\mathbf{c}(s) \neq (0,0)$. Define

$$s_1 = \sup\{s \in [0,1] | \ \varphi(s) \leq \pi\}$$

For all $s \in [0, s_1]$, $\dot{\mathbf{c}}(s)$ is in the upper half-plane. For regular curves, this implies that the arc $\mathbf{c}\big|_{[0,s_1]}$ is simple, so for $0 < s \leq s_1$,

$$\mathbf{c}(s) \neq \mathbf{c}(0) = (0,0)$$

Similarly, for all $s \in [s_1, 1]$, $\mathbf{c}(s)$ is in the lower half-plane. Therefore $\mathbf{c}\big|_{[s_1,1]}$ is simple, and for $s_1 \leq s < 1$,

$$\mathbf{c}(s) \neq \mathbf{c}(1) = (0,0)$$

References

[Csi] Csikos B.: *Differential Geometry*. Budapest Semesters in Mathematics. 1