Curves

Lagrange interpolation A set of $n+1$ points, $P_0, P_1, \cdots, P_n$, can be interpolated,

Curve $Q(t) =$ ____

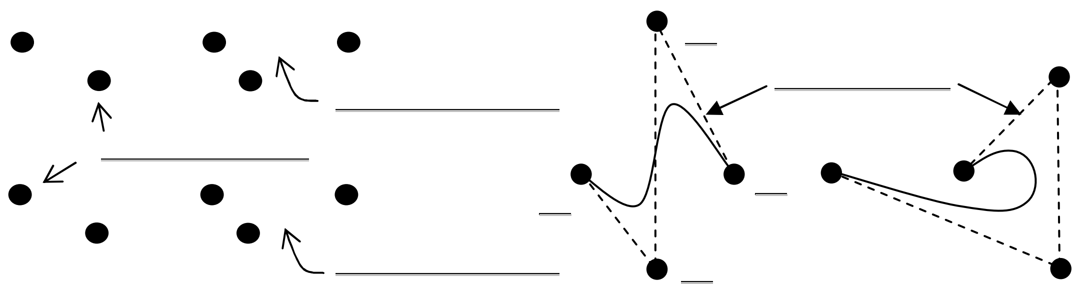
where $L_i^n(t)$ are *Lagrange polynomials of degree n*:

$L_i^n(t) =$ ____________________ $,(j \neq i)$ = ________________________________ $,(j \neq i)$.

Example: 3rd degree Lagrange polynomials (n = 3) are:

$L_i^3(t) =$ ____ $,(j \neq i)$.

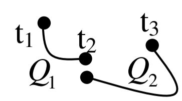
Piecewise continuous parametric polynomial curves



Continuity at the join point between segments

C^0, C^1, C^2, ... "______________" continuity. G^0, G^1, G^2, ... "______________" continuity.

Two polynomial curve segments: $Q_1(t) \quad t \in [t_1, t_2]$, $Q_2(t) \quad t \in [t_2, t_3]$



C^0, G^0: $Q_1(t_2)$ = ________ (coordinates are equal at the join point)

G^1: $Q_1'(t_2)$ = ________ (parametric 1st derivatives are *proportional* at the join point)

___: $Q_1'(t_2) = Q_2'(t_2)$ (parametric 1st derivatives (tangents) are equal at the join point)

G^2: ________ = $kQ_2''(t_2)$ (____)

C^2: ________ = ______ (____)

Parametric linear polynomial curve:

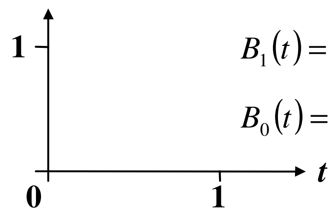
A "curve" $Q(t)$ defined by two control points, P_0 and P_1, using a *linear polynomial* ($\Rightarrow$ line).

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \begin{array}{l} x(t) = a_x t + b_x \\ y(t) = a_y t + b_y \end{array}$$

P_0 and P_1 are called the "_____________________ constraints".

One function for each control point: $B_0(t), B_1(t)$,

called "_______________________________":



$B_1(t) =$ _____

$B_0(t) =$ _____

$Q(t)$ = _______ + _______ .

Behavior of the curve at t = 0: $B_0(0)$ = _____ and $B_1(0)$ = _____ $\Rightarrow$ $Q(0)$ = _______ .

at t = 1: $B_0(1)$ = _____ and $B_1(1)$ = _____ $\Rightarrow$ $Q(1)$ = _______ .

Parametric equation for a linear polynomial curve defined by two control points:

$Q(t)$ = _______ + _______ . (… the familiar parametric equation of a _______)

A point on the line is the sum of: the ________________ (P_0, P_1)

weighted by: the _________________ $(1-t), (t)$.

$Q(t) =$ ______________

Parametric cubic polynomial curves (in general)

Cubic polynomials defining a curve segment $Q(t)=\begin{bmatrix} \quad \end{bmatrix}$ are of the form:

$x(t)=$ ____________________

$y(t)=$ ____________________, $0 \le t \le 1$

Rewrite:

$$Q(t)=\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}=\begin{bmatrix} a_x t^3 + b_x t^2 + c_x t + d_x \\ a_y t^3 + b_y t^2 + c_y t + d_y \end{bmatrix}=T\cdot C=[\qquad\qquad\qquad]\begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$$

$$\text{Let } C=M\cdot G=\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}\begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$$

(M is called the ____________.) Then $Q(t)=T\cdot M\cdot G=\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}\begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix}$

G is called the ____________, and contains 4 _________________ : $G=\begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix}=\begin{bmatrix} g_{1x} & \underline{\quad} \\ g_{2x} & \underline{\quad} \\ g_{3x} & \underline{\quad} \\ g_{4x} & \underline{\quad} \end{bmatrix}$.

Let $G_x=\begin{bmatrix} g_{1x} \\ g_{2x} \\ g_{3x} \\ g_{4x} \end{bmatrix}=\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix}_x$ (similarly for G_y). Then $x(t)=$ ____________ , and $y(t)=$ ____________ .

$$\begin{aligned} x(t)=\ &(t^3 m_{11}+t^2 m_{21}+t m_{31}+m_{41})g_{1x}+ \\ &(t^3 m_{12}+t^2 m_{22}+t m_{32}+m_{42})g_{2x}+ \\ &(t^3 m_{13}+t^2 m_{23}+t m_{33}+m_{43})g_{3x}+ \\ &(t^3 m_{14}+t^2 m_{24}+t m_{34}+m_{44})g_{4x} \end{aligned}$$

… similarly for $y(t)$

$\Rightarrow$ The curve is the ______________ of elements of the _________________ : $Q(t)=\sum_i$ ________ .

$B_i(t)$ are cubic polynomial ____________________ , given by $B=$ __________ .

Curve	Geometric constraints given by
Hermite	
Catmull-Rom	
Bezier	
B-Spline	

Cubic Hermite spline $$Q(t)=\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}=\begin{bmatrix} a_x t^3 + b_x t^2 + c_x t + d_x \\ a_y t^3 + b_y t^2 + c_y t + d_y \end{bmatrix}=\sum_i B_i(t) G_i \ = \ T \cdot \_\_\_ \cdot \_\_\_ .$$

Shape is determined by:

- two ______________________________, ___ and ___ .
- two ______________________________, ___ and ___ .

$$G_{\mathrm{H}} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}, \ G_{\mathrm{H}_x} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}_x, \ G_{\mathrm{H}_y} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}_y$$

M_H is the ____________________ that relates the ____________________, G_{H}, to the cubic polynomial coefficients (a_x, b_x, …, a_y, b_y, …).

To find M_H : for each geometric constraint we ______________________________ , then solve for the unknowns.

$$x(t) = T \cdot M_H \cdot G_{H_x} = a_x t^3 + b_x t^2 + c_x t + d_x$$

$$x(t) = [\qquad\qquad\qquad\qquad] \cdot M_H \cdot G_{H_x}$$

$$x'(t) = [\qquad\qquad\qquad\qquad] \cdot M_H \cdot G_{H_x}$$

Endpoint constraints are given by:

$$x(0) = P_{1_x} \quad = [\qquad\qquad\qquad] \cdot M_H \cdot G_{H_x}$$
$$x(1) = P_{4_x} \quad = [\qquad\qquad\qquad] \cdot M_H \cdot G_{H_x}$$

Tangent vector constraints are given by:

$$x'(0) = R_{1_x} \quad = [\qquad\qquad\qquad] \cdot M_H \cdot G_{H_x}$$
$$x'(1) = R_{4_x} \quad = [\qquad\qquad\qquad] \cdot M_H \cdot G_{H_x}$$

Rewrite in matrix form: $G_{\mathrm{H}_x} = \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix}_x = \begin{bmatrix} \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \end{bmatrix} \cdot M_{\mathrm{H}} \cdot G_{\mathrm{H}_x}$

$$\Rightarrow M_{\mathrm{H}} = \begin{bmatrix} \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \end{bmatrix}^{-1} = \begin{bmatrix} \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \end{bmatrix}.$$

Use G_{H} and M_{H} to solve $Q(t) = T \cdot M_H \cdot G_H = [x(t) \quad y(t)]$:

$$Q(t) = T \cdot M_{\mathrm{H}} \cdot G_{\mathrm{H}} = [\qquad\qquad\qquad\qquad] \cdot \begin{bmatrix} \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \end{bmatrix} \cdot \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \\ \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} T \cdot M_{\mathrm{H}} \cdot G_{\mathrm{H}_x} \\ T \cdot M_{\mathrm{H}} \cdot G_{\mathrm{H}_y} \end{bmatrix}.$$

The cubic Hermite ______________________________, B_{H}, given by $B_{\mathrm{H}} =$ ____________ ,
are the polynomials weighting the elements of the geometry vector:

$$\begin{aligned} Q(t) &= T \cdot M_{\mathrm{H}} \cdot G_{\mathrm{H}} \\ &= B_{\mathrm{H}} \cdot G_{\mathrm{H}} \\ &= (\underline{\qquad\qquad})P_1 + (\underline{\qquad\qquad})P_4 + (\underline{\qquad\qquad})R_1 + (\underline{\qquad\qquad})R_4 \\ &= \sum_{i=0}^{3} B_{\mathrm{H}_i}(t) P_i \end{aligned}$$

$B_{H_0}(t) =$ __________

$B_{H_1}(t) =$ __________

$B_{H_2}(t) =$ __________

$B_{H_3}(t) =$ __________

Cubic Bezier spline

Shape is determined by: • two ______________________________________, ___ and ___ .

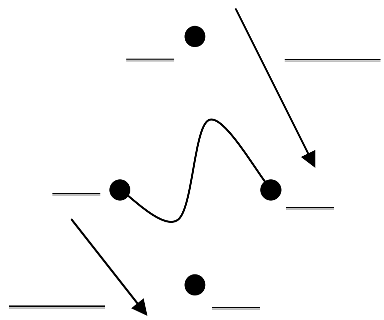
• two ______________________________________, ___ and ___ .

Geometry vector: ______________________ . Endpoint tangent vectors: _______, and _______ .

A cubic Bezier curve ______________the 2 endpts and _________________the intermediate 2 points:

The cubic Bezier endpoint tangents are related to the cubic Hermite endpoint tangents:

$$G_{\mathrm{B}} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$



$R_1 =$ ___________

$R_4 =$ ___________

M_{HB}, relates ___________________________ to ____________________________:

$$G_{\mathrm{H}} = \begin{bmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{bmatrix} = M_{\mathrm{HB}} \cdot G_{\mathrm{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{bmatrix}$$

Hermite basis: $Q(t) = T \cdot M_{\mathrm{H}} \cdot G_{\mathrm{H}}$

Substitute $M_{\mathrm{HB}} \cdot G_{\mathrm{B}}$ for G_{H}: $Q(t) = T \cdot M_{\mathrm{H}} \cdot (M_{\mathrm{HB}} \cdot G_{\mathrm{B}})$

$Q(t) = T \cdot (M_{\mathrm{H}} \cdot M_{\mathrm{HB}}) \cdot G_{\mathrm{B}}$

$M_{\mathrm{H}} \cdot M_{\mathrm{HB}} \Rightarrow M_{\mathrm{B}}$, the ______________________

$$M_{\mathrm{H}} \cdot M_{\mathrm{HB}} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix} = M_{\mathrm{B}}$$

$T \cdot M_{\mathrm{B}} \Rightarrow B_{\mathrm{B}}$. the ______________________ .

$$\begin{aligned} Q(t) &= T \cdot M_{\mathrm{B}} \cdot G_{\mathrm{B}} = B_{\mathrm{B}} \cdot G_{\mathrm{B}} \\ &= (\_\_\_\_\_\_\_\_\_)P_1 + (\_\_\_\_\_\_\_\_\_)P_2 + (\_\_\_\_\_\_\_\_\_)P_3 + (\_\_\_\_\_\_\_\_\_)P_4 \\ &= \sum_{i=0}^{3} B_{\mathrm{B}_i}(t) P_i \end{aligned}$$

$B_{\mathrm{B}_0}(t) =$ ________ $B_{\mathrm{B}_1}(t) =$ ________ $B_{\mathrm{B}_2}(t) =$ ________ $B_{\mathrm{B}_3}(t) =$ ________

Cubic B-spline:

Approximates _______ control points with a curve composed of _______ segments, where _______ . Each segment is defined by _______ control points ($n = 3 \Rightarrow$ minimum of _______ control points and _______ segment).

$$G_{BS} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}, \ M_{BS} = \frac{1}{6}\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}, \ Q(t) = T \cdot M_{BS} \cdot G_{BS}$$

$Q(t) = \frac{1}{6}($ ______ $)P_1 + ($ ______________ $)P_2 + ($ _____________________ $)P_3 + ($ __ $)P_4$

$B_{BS_0}(t) =$ ________

$B_{BS_1}(t) =$ ________________

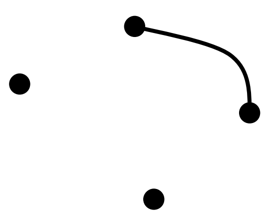
$B_{BS_2}(t) =$ _______________________

$B_{BS_3}(t) =$ ____

Cubic Cardinal spline

Shape is determined by:

- two ____, ___ and ___ .
- two ____, ___ and ___ .
- a _________________ parameter.



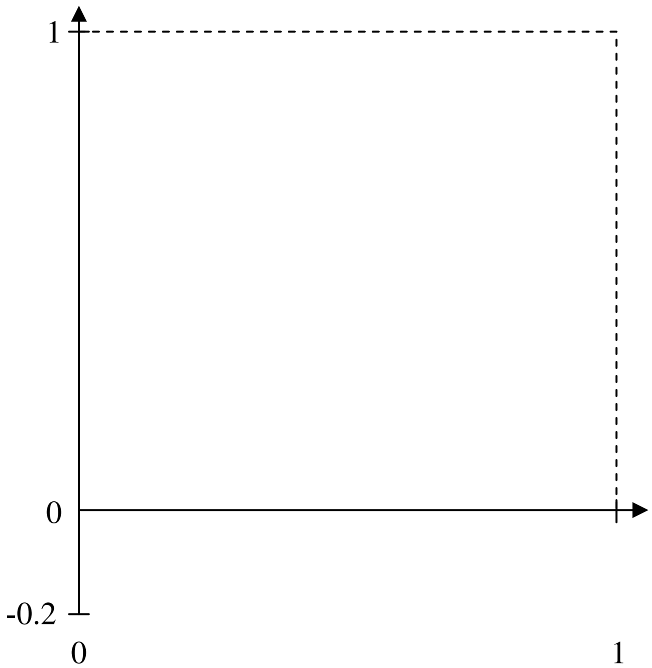
Tangent at P_2: _________________

Tangent at P_3: ________________ , where a is called the ___________ .

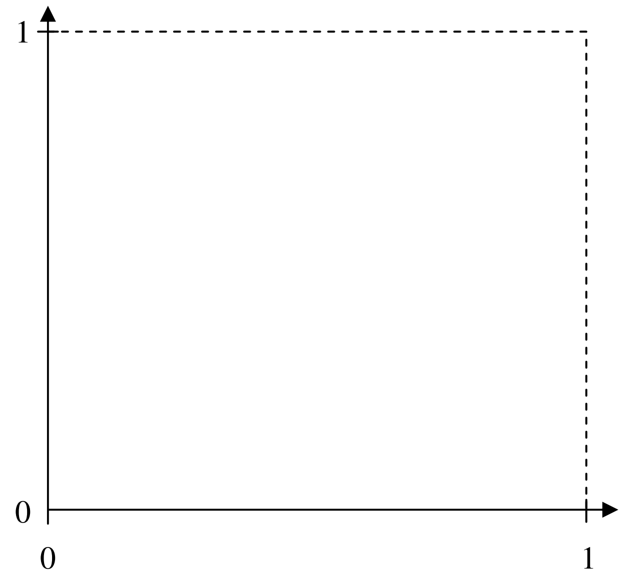
Define: $\tau =$ ___________

$$M_{CD} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}, \quad B_i(t) = T \cdot M_{CD}$$

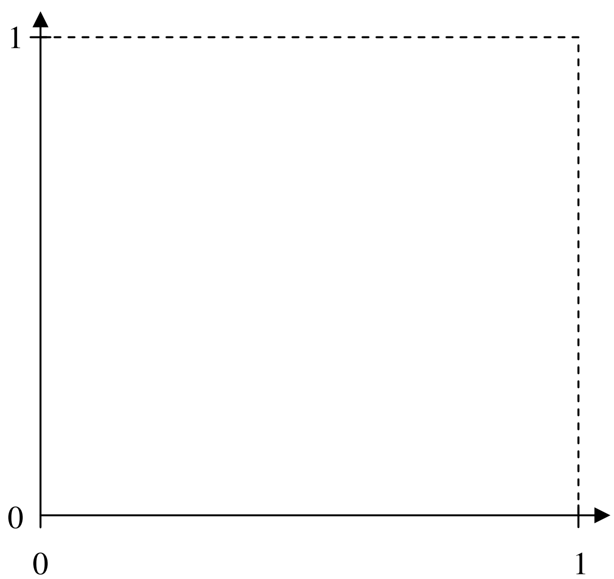
When $a = 0$ the spline is a ________________________ .



Cubic Hermite Blending Functions



Cubic Bezier Blending Functions



Cubic B-spline Blending Functions