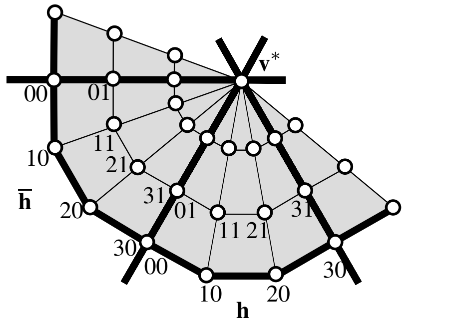

Figure 14: C^1 ***transition*** *between adjacent polar patches.*

Appendix A: Verification of smoothness

Here we formally verify the continuity constraints satisfied by the formulas of Sections 3 and 5. We first show that P_m-patches are internally C^1 and then show smoothness across patch boundaries.

Internal P_m-patch smoothness

Theorem 1 Adjacent sectors $\mathbf{b}^{i-1}$ and $\mathbf{b}^i$ of a P_m-patch meet with C^1 continuity.

Proof Two adjacent triangular patches $\mathbf{b}^{i-1}$ and $\mathbf{b}^i$, as labeled in Figure 4, meet C^0 since the control points along their shared boundary are equal by (3) and they satisfy the C^1 constraints [Far90, p.104] with k_1 and k_2 defined in (3) for $l = 0,1,2$. To complete the proof, it remains to show that (3) also holds for $l = 3$, i.e. establish C^1 continuity (5) at the central point.
If $m = 4$ then $k_1 = 0$ and $k_2 = 1/2$ and (3) specializes to $\mathbf{b}^i_{103} := \frac{1}{2}\left(\mathbf{b}^i_{112} + \mathbf{b}^{i-1}_{112}\right)$ so that

$$\left(\mathbf{b}^{i+1}_{103} + \mathbf{b}^{i-1}_{103}\right)/2 = \frac{1}{4}\sum_{i=0}^{3}\mathbf{b}^i_{112}.$$

Substituting $\mathbf{b}^i_{112}$ from (21) results in cancellation of all terms with subscripts 121 and 211, leaving $\left(\mathbf{b}^{i+1}_{103} + \mathbf{b}^{i-1}_{103}\right)/2 = \mathbf{b}_{004}$, i.e. (5) for $m = 4$.
If $m = 3$ then $i-2 = i+1$, $k_2 = 1/3 = k_1$ and (3) for $l = 2$ yields $\mathbf{b}^i_{103} = (\mathbf{b}^i_{202} + \mathbf{b}^i_{112} + \mathbf{b}^{i-1}_{112})/3$ so that

$$3(\mathbf{b}^{i+1}_{103} + \mathbf{b}^{i-1}_{103} + \mathbf{b}^i_{103}) = \mathbf{b}^{i+1}_{202} + \mathbf{b}^{i-1}_{202} + \mathbf{b}^i_{202} + 2(\mathbf{b}^{i+1}_{112} + \mathbf{b}^{i-1}_{112} + \mathbf{b}^i_{112}).$$

Substitution of $\mathbf{b}^i_{112} = 3\mathbf{b}_{004}/2 - \mathbf{b}^{i+2}_{202}/2$ according to (21) yields (5) in the form

$$\mathbf{b}^{i+1}_{103} + \mathbf{b}^{i-1}_{103} + \mathbf{b}^i_{103} = 3\mathbf{b}_{004}.$$

The final case, $m = 5$ is checked analogously using symbolic substitution and the relation $\mathsf{c}_m = -\frac{k_1}{2k_2}$. □

Smoothness across patches

Adjacent ordinary patches meet with C^2 continuity since we reproduce the standard bi-cubic B-spline in this case. Since the shared cubic boundary of two adjacent patches (P_m-patches, polar, or ordinary) is defined by unique coefficients $(\mathbf{v}^0, \mathbf{t}^0_0, \mathbf{t}^1_1, \mathbf{v}^1)$, C^0 continuity is guaranteed and bicubic ordinary patches (resp. polar patches) and P_m-patches match up exactly, and we need only verify tangent continuity across.

Theorem 2 The surface corresponding to a polar configuration is C^1.

Proof Consider two adjacent polar patches $\mathbf{h}$ and $\overline{\mathbf{h}}$ with $\overline{\mathbf{h}}_{3i} = \mathbf{h}_{0i}$ for $i = 0,1,2,3$ (Figure 14). We need only show that

$$\overline{\mathbf{h}}_{3i} = \mathbf{h}_{0i} = \frac{1}{2}\left(\overline{\mathbf{h}}_{2i} + \mathbf{h}_{1i}\right). \quad (23)$$

and that there exists a unique normal at $\mathbf{v}^* := \mathbf{h}_{03}$, the point of parametric singularity. By specializing Section 5.1 to the valence $n = 4$ at $\overline{\mathbf{h}}_{30} = \mathbf{h}_{00}$ this condition is satisfied for $i = 0,1$. It is trivially satisfied for $i = 3$ due to the singularity at $\mathbf{v}^*$. Since $(\overline{\mathbf{h}}_{02} - \mathbf{v}^*) + (\mathbf{h}_{32} - \mathbf{v}^*) = 2\mathsf{c}_n(\mathbf{h}_{02} - \mathbf{v}^*)$ due to (16), substituting (18) and letting n be the valence at $\mathbf{v}^*$ yields

$$\begin{aligned}
&\frac{1}{2}\left(\overline{\mathbf{h}}_{22} + \mathbf{h}_{12}\right)\\
&= \frac{1}{2}\left(\frac{2\overline{\mathbf{h}}_{32} + \overline{\mathbf{h}}_{02} + (\mathsf{c}_n - 1)\mathbf{v}^*}{2 + \mathsf{c}_n} + \frac{2\mathbf{h}_{02} + \mathbf{h}_{32} + (\mathsf{c}_n - 1)\mathbf{v}^*}{2 + \mathsf{c}_n}\right)\\
&= \frac{1}{2(2+\mathsf{c}_n)}\left(4\mathbf{h}_{02} + 2\mathsf{c}_n(\mathbf{h}_{02} - \mathbf{v}^*) + 2\mathbf{v}^* + 2(\mathsf{c}_n - 1)\mathbf{v}^*\right)\\
&= \mathbf{h}_{02},
\end{aligned}$$

satisfying (23) for $i = 2$ as well.
Since all the vectors $\{\mathbf{h}_{i2} - \mathbf{v}^*\}_{i=0,1,2,3}$ and $\{\overline{\mathbf{h}}_{i2} - \mathbf{v}^*\}_{i=0,1,2,3}$ are co-planar by construction (16), (18), they define a unique tangent plane and normal at $\mathbf{v}^*$.
Crease formula (19) uniformly scales $\overline{\mathbf{h}}_{22}$ and $\mathbf{h}_{12}$ toward $\mathbf{h}_{02}$ without affecting their average, still satisfying the C^1 constraints. □

Since the polar construction (Figure 11) is identical to the ordinary case for the two layers of Bézier control points away from the polar vertex – $\mathbf{h}_{i0}$ and $\mathbf{h}_{i1}$ in Figure 14 – the polar patch behaves identically to ordinary patches in regard to first order continuity along the boundary $\mathbf{h}_{00}$—$\mathbf{h}_{30}$. In particular, one can easily verify that these patches meet C^1 with ordinary patches using the simplified formulas for $n = 4$ at the end of Section 5.1.

Theorem 3 P_m-patches meet G^1 with ordinary patches (resp. polar patches) and other P_m-patches.

Proof First, we consider the case where a P_m-patch $\mathbf{b}^i$ and an ordinary patch (resp. polar patch) $\mathbf{a}$ share a boundary $\mathbf{b}^i(u,0) = \mathbf{a}(0,u)$. In this case the valences at the endpoints are both $n^0 = 4 = n^1$ and therefore $\mathsf{c}_{n^0} = \mathsf{c}_{n^1} = 0$, $\mathsf{s}_{n^0} = \mathsf{s}_{n^1} =$

1, $\lambda_n = 1/2$, $\omega_n = 4$, $\sigma_n = 1/2$ and hence $\mathbf{t}^i_j = \mathbf{e}^i_j$. We show that the following G^1 condition holds:

$$(1-2c_m u)\partial_1 \mathbf{b}^i(u,0) = 2\mu\partial_2 \mathbf{b}^i(u,0) + \partial_1 \mathbf{a}(0,u) \qquad (11)$$

where $\mu := 1 - c_m$ is defined in (20) and depends on the m, the number of sides of the P_m-patch. By setting two polynomials of maximal degree 3 equal, equation (11) holds if and only if the four coefficients are equal. We need only verify equality of the first two since the other two can be equally verified by starting from the other endpoint. The first coefficient equation (11_1) establishes that the per-vertex computation generates a common tangent plane at the endpoint $u=0$: $\partial_1\mathbf{b}^i(0,0) = 2\mu\partial_2\mathbf{b}^i(0,0) + \partial_1\mathbf{a}(0,0)$. In terms of coefficients (see Figure 15), since $8\mu k_2 = 4$,

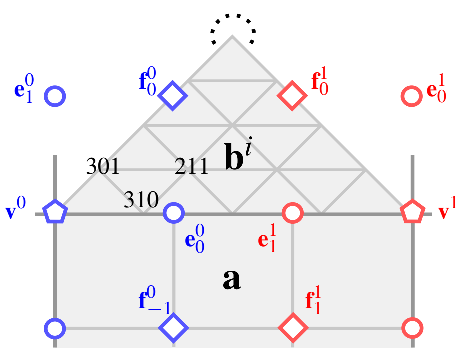

Figure 15: G^1 ***transition*** *between a P_m-patch $\mathbf{b}^i$ and a bicubic patch* **a**.

$$2\mu\cdot 4(\mathbf{b}^i_{301} - \mathbf{v}^0) = 8\mu\left((k_1-1)\mathbf{v}^0 + k_2(\mathbf{b}^i_{310} + \mathbf{b}^{i-1}_{130})\right)$$
$$= 4\left(\frac{3}{4}(\mathbf{e}^0_0 + \mathbf{e}^1_0 - 2\mathbf{v}^0)\right) = 3(\mathbf{e}^0_0 + \mathbf{e}^1_0) - 6v^0.$$

Therefore equation (11_1) simplifies to and is easily verified as

$$\underbrace{3(\mathbf{e}^0_0 - \mathbf{v}^0)}_{\partial_1\mathbf{b}^i(0,0)} = \underbrace{3(\mathbf{e}^0_0 + \mathbf{e}^0_1) - 6\mathbf{v}^0}_{2\mu\partial_2\mathbf{b}^i(0,0)} \underbrace{-3(\mathbf{e}^0_1 - \mathbf{v}^0)}_{\partial_1\mathbf{a}(0,0)} \qquad (11_1)$$

The second BB-coefficient of $(1-2c_m u)\partial_1\mathbf{b}^i(u,0)$ is

$$3\frac{(2\mu-1)(\mathbf{e}^0_0 - \mathbf{v}_0) + 2(\mathbf{e}^1_1 - \mathbf{e}^0_0)}{3}$$

and of $2\mu\partial_2\mathbf{b}^i(u,0)$ is (cf. Figure 15)

$$2\mu\cdot 4(\mathbf{b}^i_{211} - \mathbf{b}^i_{310})$$
$$= 8\mu\left(\frac{\mathbf{e}^1_1 - \mathbf{e}^0_0}{4\mu} + \frac{2\mu-1}{8\mu}(\mathbf{e}^0_0 - \mathbf{v}^0) + 3\frac{\mathbf{f}^0 - \mathbf{e}^0_0}{8\mu}\right)$$
$$= 2(\mathbf{e}^1_1 - \mathbf{e}^0_0) + (2\mu-1)(\mathbf{e}^0_0 - \mathbf{v}^0) + 3(\mathbf{f}^0 - \mathbf{e}^0_0).$$

and so the second equality simplifies to and is easily checked as

$$(2\mu-1)(\mathbf{e}^0_0 - \mathbf{v}^0) + 2(\mathbf{e}^1_0 - \mathbf{e}^0_0) = 2(\mathbf{e}^0_1 - \mathbf{e}^0_0) \qquad (11_2)$$
$$+ (2\mu-1)(\mathbf{e}^0_0 - \mathbf{v}^0) + 3(\mathbf{f}^0 - \mathbf{e}^0_0) - 3(\mathbf{f}^0 - \mathbf{e}^0_0).$$

If both facets are extraordinary with $m_\mathbf{b}$ and $m_\mathbf{a}$ sides respectively then with

$$\ell_0 := \xi^0,\ \ell_1 := 2\mu - \xi^1,\ \xi^i := 1 + c_{n^i}$$
$$\ell(u) := \left((1-u)\ell_0 + u\ell_1\right),\ \mu := 1 - c_{m_\mathbf{b}},\ \nu := 1 - c_{m_\mathbf{a}}$$

the G^1 constraints on the two corresponding sector patches **b** and **a** are

$$\ell(u)\partial_1\mathbf{b}(u,0) = \mu\partial_2\mathbf{b}(u,0) + \nu\partial_1\mathbf{a}(0,u). \qquad (12)$$

Again we need only verify the first two coefficient equation (12_1) and (12_2) of the four equivalent to (12). By (3) and

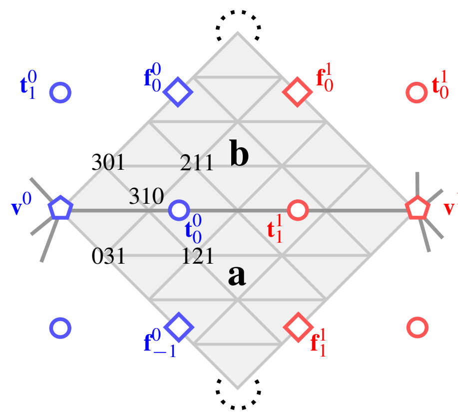

Figure 16: G^1 ***transition*** *between sectors of a $P_{m_\mathbf{b}}$-patch* **b** *and a $P_{m_\mathbf{a}}$-patch* **a**.

since $k_1 = \frac{\mu-1}{\mu}$ and $k_2 = \frac{1}{2\mu}$,

$$\mu\partial_2\mathbf{b}(0,0) = 4\mu\left(\frac{\mu-1}{\mu}\mathbf{v}^0 + \frac{1}{2\mu}(\mathbf{b}^i_{310} + \mathbf{b}^{i-1}_{130})\right)$$
$$= 2(\mathbf{b}^i_{310} + \mathbf{b}^{i-1}_{130} - 2\mathbf{v}^0) = 2\frac{3}{4}(\mathbf{t}^0_1 + \mathbf{t}^0_0 - 2\mathbf{v}^0).$$

Since $(\mathbf{t}^0_{-1} - \mathbf{v}^0) + (\mathbf{t}^0_1 - \mathbf{v}^0) = 2c_{n^0}(\mathbf{t}^0_0 - \mathbf{v}^0)$, (12_1) simplifies to and is easily checked as

$$\underbrace{3(1+c_{n^0})(\mathbf{t}^0_0 - \mathbf{v}^0)}_{\ell_0\partial_1\mathbf{b}(0,0)} = \underbrace{3\left(\frac{\mathbf{t}^0_1 + \mathbf{t}^0_0}{2} - \mathbf{v}^0\right)}_{\mu\partial_2\mathbf{b}(0,0)} + \underbrace{3\left(\frac{\mathbf{t}^0_{-1} + \mathbf{t}^0_0}{2} - \mathbf{v}^0\right)}_{\nu\partial_1\mathbf{a}(0,0)}$$
$$= 3\frac{1}{2}\left(2c_{n^0}(\mathbf{t}^0_0 - \mathbf{v}^0) + 2(\mathbf{t}^0_0 - \mathbf{v}^0)\right). \qquad (12_1)$$

Since the transversal terms involving $\mathbf{f}^0_0 - \mathbf{f}^{-1}_0$ in the expansions of $\mu\mathbf{b}_{211}$ and $\nu\mathbf{a}_{121}$ cancel, the second coefficient equation simplifies to and is easily checked as

$$6\ell_0(\mathbf{t}^1_1 - \mathbf{t}^0_0) + 3\ell_1(\mathbf{t}^0_0 - \mathbf{v}^0)$$
$$= 12\mu(\mathbf{b}_{211} - \mathbf{b}_{310}) + 12\nu(\mathbf{a}_{121} - \mathbf{b}_{310})$$
$$= 6\xi^0(\mathbf{t}^1_1 - \mathbf{t}^0_0) + 3(2\mu - \xi^1)(\mathbf{t}^0_0 - \mathbf{v}^0). \qquad (12_2)$$

So the claim of smoothness is verified. □

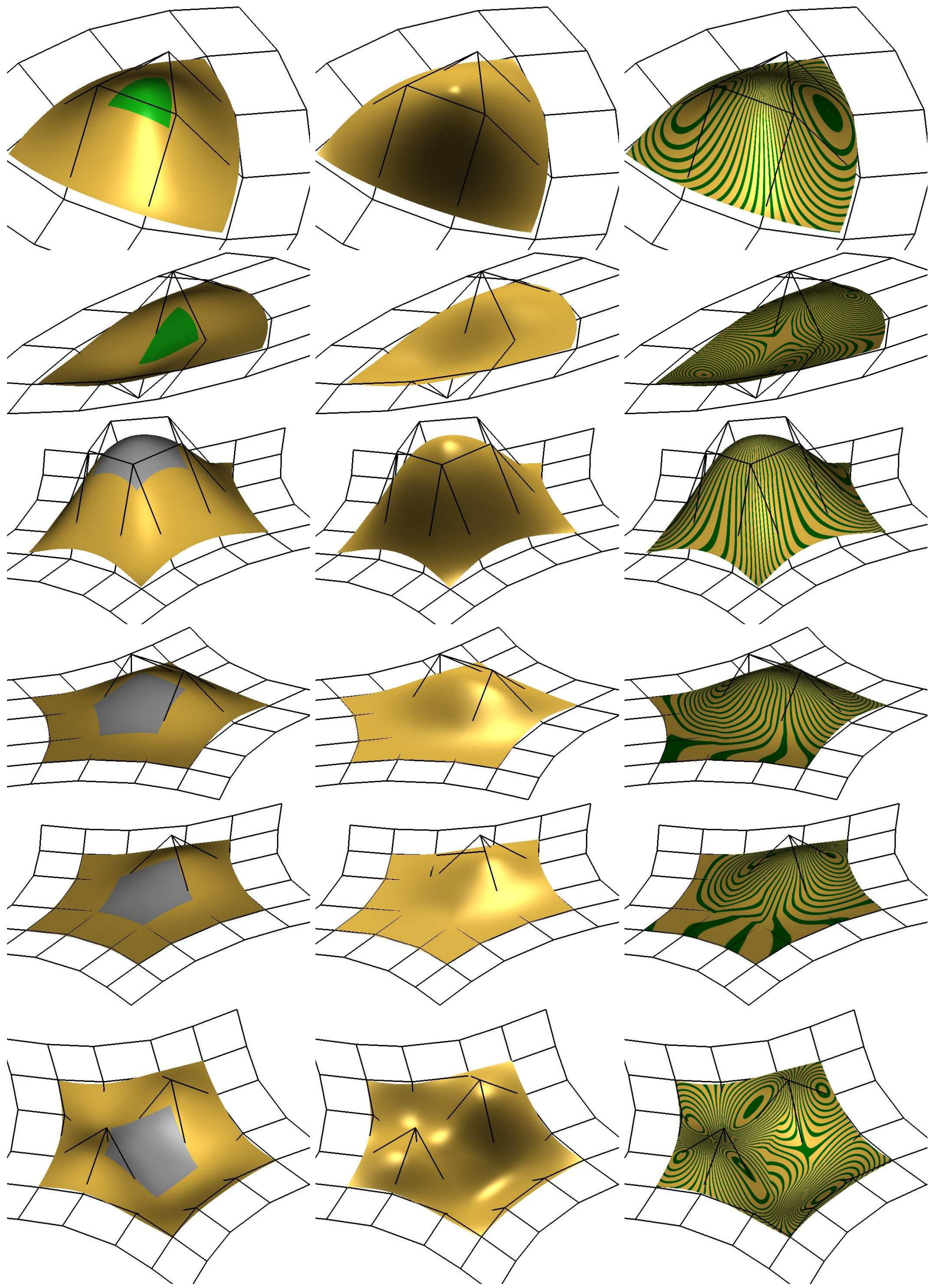

Figure 17: ***Elliptic and saddle shapes with*** P_3***-patch (*****top two rows*****) and*** P_5***-patch (*****rest*****).**

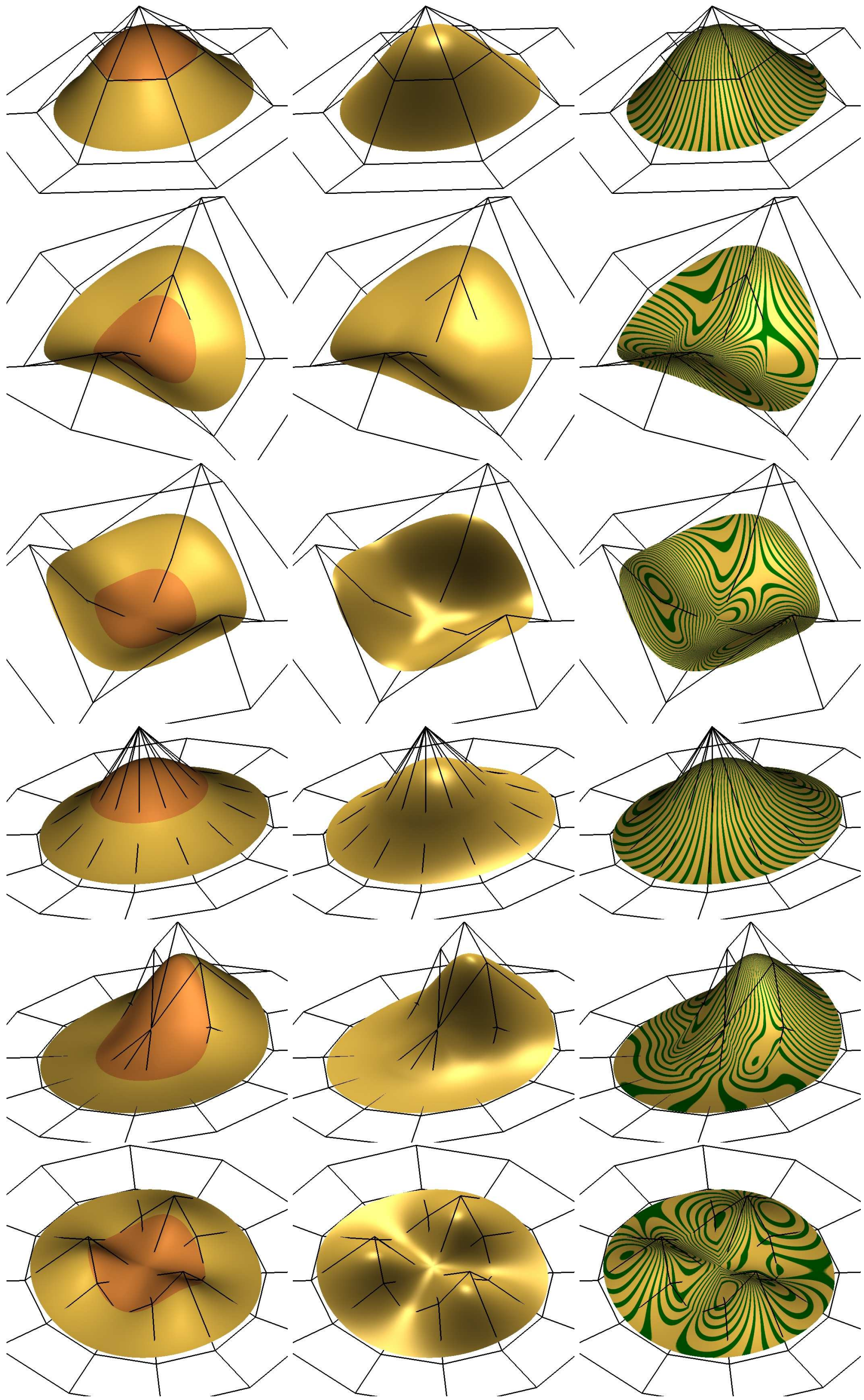

Figure 18: ***Elliptic and saddle shapes with 6- (*****top three rows*****) and 12-valent (*****rest*****) polar configurations.***