Spectral methods in deformable shape analysis

Alex Bronstein, Michael Bronstein, Umberto Castellani

March 14, 2012

Dimensions of media

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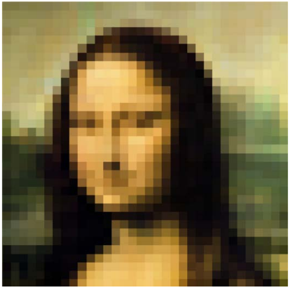
3D shapes vs. images



Array of pixels

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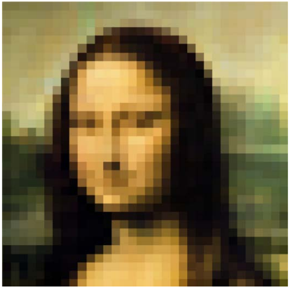

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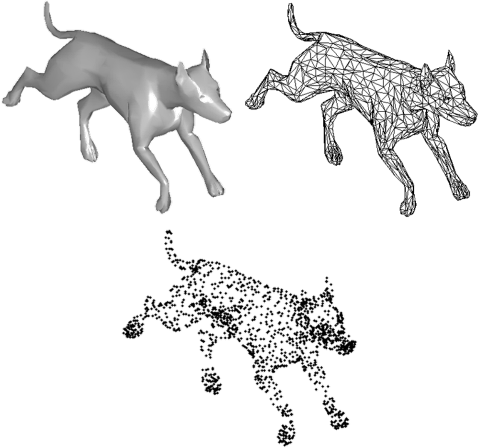

Splines, Mesh, Point cloud, etc

3D shapes vs. images

Affine, projective

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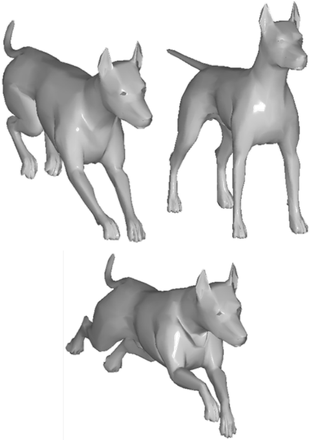
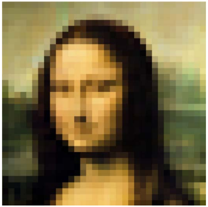
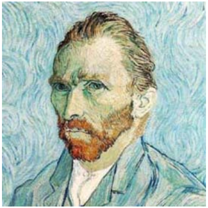


Wealth of nonrigid deformations

3D shapes vs. images

$\frac{1}{2}$  $+\frac{1}{2}$  $=$ 

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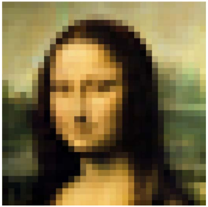

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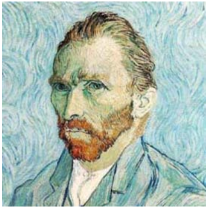

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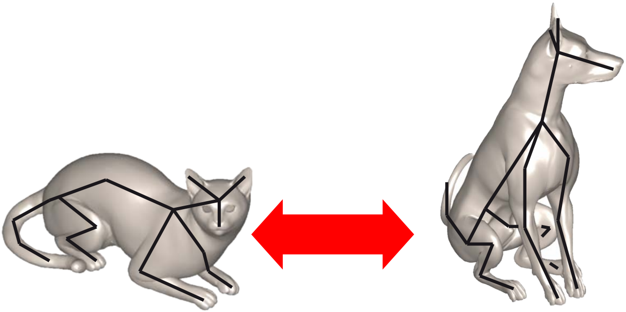
Setting



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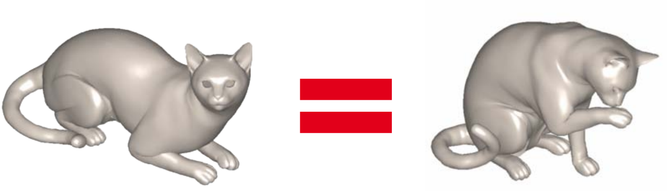

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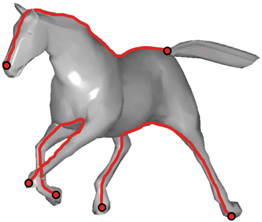
- Similarity, correspondence, retrieval, etc. = similarity and correspondence between structures
- *Invariance* under bending, scale, affine transformations, etc.

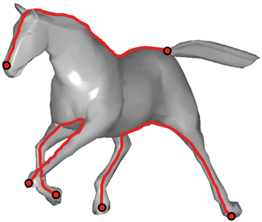
Structure



Global structure

Metric space

Structure



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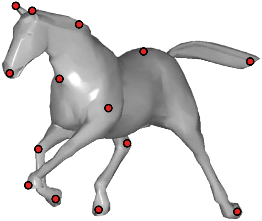

Local structure
Point descriptors

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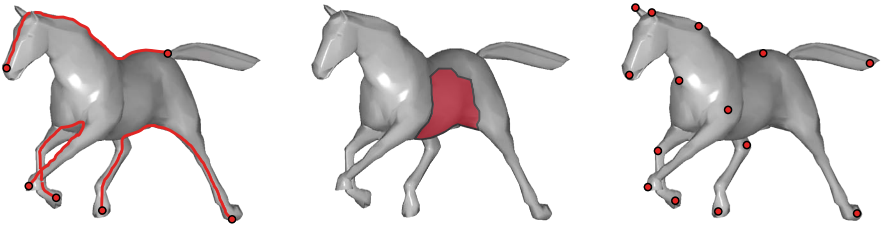

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Glocal structure
Stable regions

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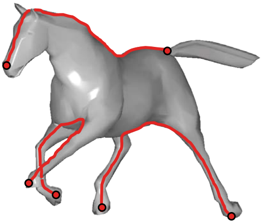

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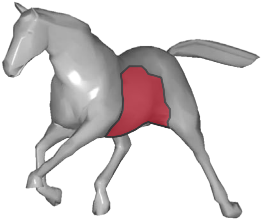

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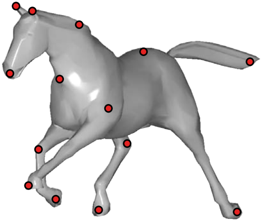

Local structure
Point descriptors

Agenda

- Diffusion processes on surfaces
- Spectral point of view
- *Global structure*: diffusion geometry
- *Local structure*: diffusion kernel descriptors
- *Semi-local structure*: maximally stable components
- Extensions

Diffusion processes on surfaces

- *Heat equation*

$$\left(\Delta_X + \frac{\partial}{\partial t}\right) u = 0$$

governs heat propagation on manifold X

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- *Solution* $u(x,t)$: heat distribution at point x at time t
- *Initial condition* $u_0(x)$: heat distribution at time $t = 0$
- *Boundary condition* if manifold has a boundary

Laplace-Beltrami operator Δ_X

For two smooth functions $f, g : X \to \mathbb{R}$ and standard inner product on X

$$\langle f, g \rangle = \int_X f(x)g(x)\, da$$

the Laplacian satisfies the following properties:

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- *Maximum principle*: functions satisfying $\Delta_X f = 0$ (harmonic) have no minima/maxima in the interior of X

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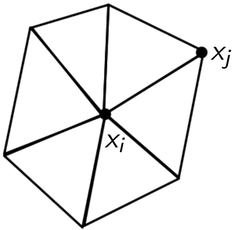
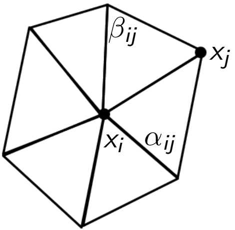
- In matrix notation

$$L_X f = A^{-1} L f$$

where $A = \mathrm{diag}\{a_i\}$ and $(L)_{ij} = \mathrm{diag}\left\{\sum_{k \neq i} w_{ik}\right\} - w_{ij}$

Discretization of the Laplacian





Discrete Laplacian

$$w_{ij} = \begin{cases} 1 & : x_j \in \mathcal{N}_1(x_i) \\ 0 & : \text{else} \end{cases}$$

$a_i = 1$ (umbrella operator); or
$a_i = |\mathcal{N}_1(x_i)|$, valence (Tutte)

Discretized Laplacian

$$w_{ij} = \begin{cases} \cot \alpha_{ij} + \cot \beta_{ij} & : x_j \in \mathcal{N}_1(x_i) \\ 0 & : \text{else} \end{cases}$$

$a_i =$ sum of areas of triangles sharing vertex x_i

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  Reason: Flat plate must have zero bending energy.

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  Indispensable for discretization of PDE solutions

No free lunch

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- Many attempts have been made to construct discrete Laplacians satisfying above desired properties
- **"No free lunch theorem" (Wardetzky et al., 2007)**
  There is no discrete Laplacian satisfying the above properties simultaneously!

Eigendecomposition of Laplacian

- On *compact* domains Laplacian admits *countable orthogonal eigendecomposition*

$$\Delta_X \phi_i = \lambda_i \phi_i$$

λ_i – eigenvalues; $\phi_i(x)$ – corresponding eigenfunctions

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- Discrete *generalized eigendecomposition* for $L_X = A^{-1}L$

$$A\Phi = \Lambda L \Phi$$

  $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_k\}$ – diagonal matrix of first k eigenvalues
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- *Spectral decomposition theorem*

$$(\Delta_X f)(x) = \sum_{i \geq 0} \lambda_i \phi_i(x) \cdot \langle \phi_i, f \rangle$$

Discrete equivalent: $L_X f = \sum_{i \geq 0} \lambda_i \phi_i \phi_i^{\mathrm{T}} f$

Finite elements

- Discretize $\{\lambda_i, \phi_i\}$ directly!

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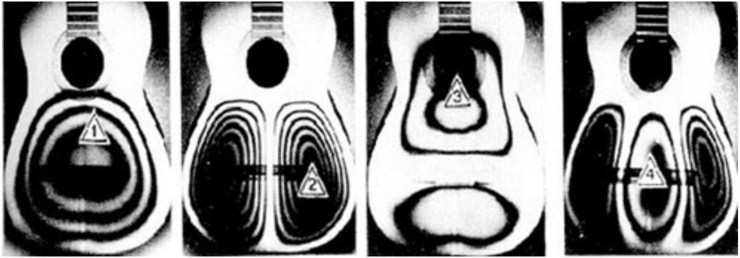
- In matrix notation: $Au = \lambda Bu$

To see the sound

Chladni plates



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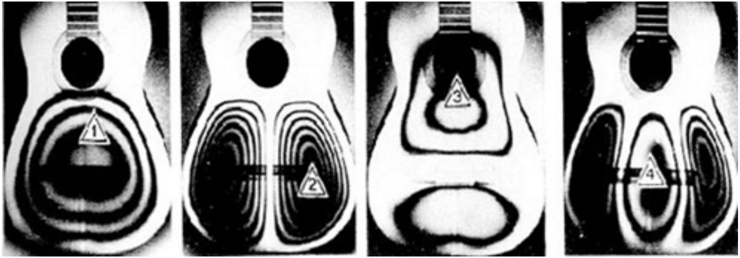

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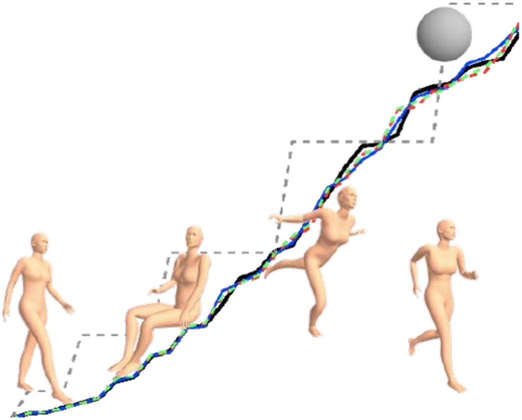
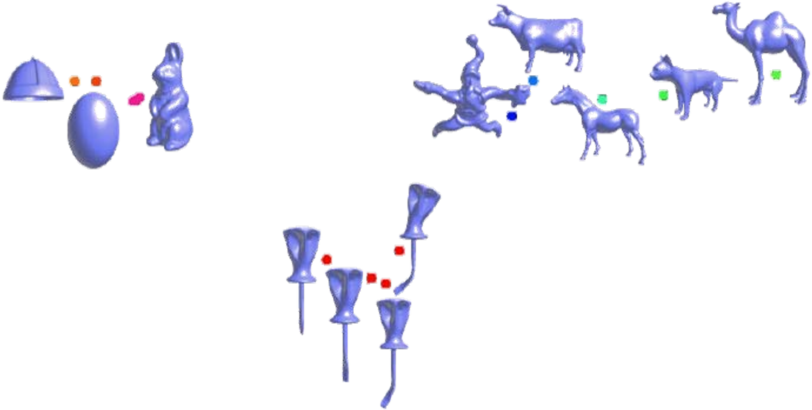
- Laplacian *eigenfunctions* = plate vibration modes

Shape DNA



(Reuter *et al.*, 2006) use Laplacian spectrum $\{\lambda_i\}$ as an isometry-invariant shape descriptor – *shape DNA*

Shape DNA



Shape similarity using Shape DNA

“Can we hear the shape of the drum?”

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- Are isospectral shapes isometric?
- **Can one hear the shape of the drum?** (*Mark Kac*)

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The following shape properties can be recovered (“heard”) from the spectrum of the Laplacian:

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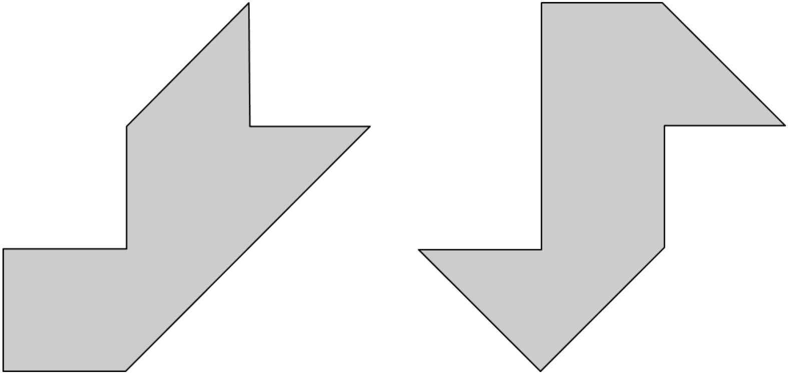
- Total Gaussian curvature

“Can we hear the shape of the drum?”

The following shape properties can be recovered (“heard”) from the spectrum of the Laplacian:

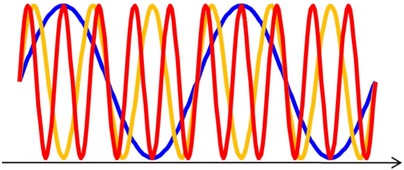
- Area
- Euler characteristic (genus)
- Total Gaussian curvature
- **Can we hear the metric?**

One cannot hear the shape of the drum!



Counter example of isospectral non-isometric shapes
(Gordon *et al.*, 1991)

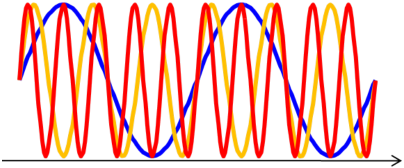
Relation to harmonic analysis



1D signals

$$-\frac{d^2}{dx^2}e^{inx} = n^2 e^{inx}$$

Relation to harmonic analysis



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3D shapes

$$\Delta_X \phi_i(x) = \lambda_i \phi_i(x)$$

Relation to harmonic analysis

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- $\{F(\lambda_i)\} =$ *Fourier coefficients*

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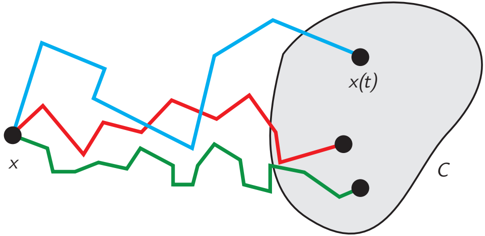
- Heat operator can be interpreted as a non shift-invariant version of convolution

Heat kernel

Probabilistic interpretation

- *Brownian motion* $x(t)$ starts at point x



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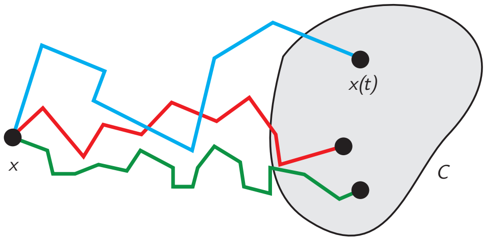

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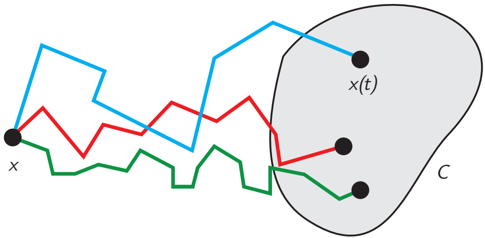

- $\Pr(x(t) \in C) = \int_C h_t(x,y)da(y)$
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Diffusion geometry

- Family of *diffusion metrics*

$$\begin{aligned} d^2(x,y) &= \|k(x,\cdot) - k(y,\cdot)\|^2_{L^2(X)} \\ &= \int_X (k(x,z) - k(y,z))^2 da(z) \end{aligned}$$

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- "Connectability" by random walks of any length
- **Scale invariant!**

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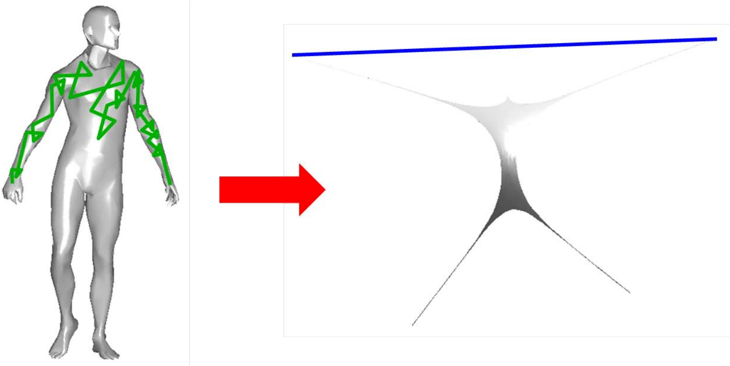
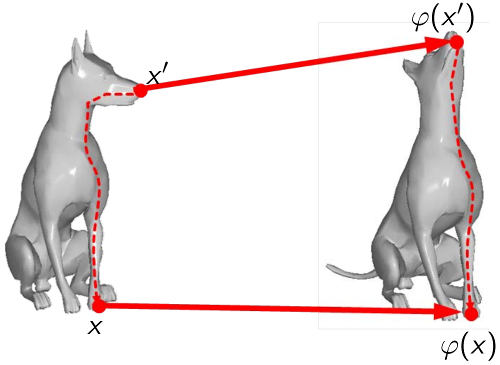
- Diffusion distance is represented by *Euclidean distance* in embedding space

Diffusion maps

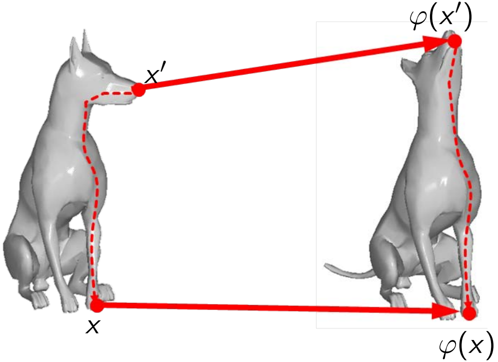


Correspondence



- Embed one shape into the other

Correspondence



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- Find *minimum distortion* correspondence

$$\min_{\varphi:X\to Y} \|d_X - d_Y \circ (\varphi \times \varphi)\|$$

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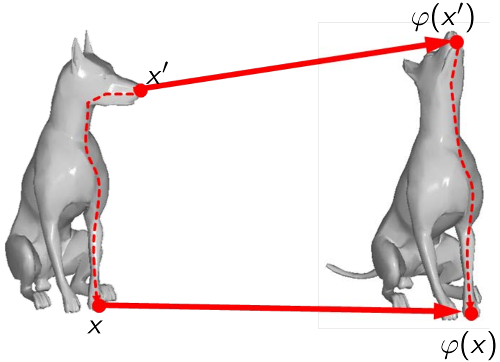

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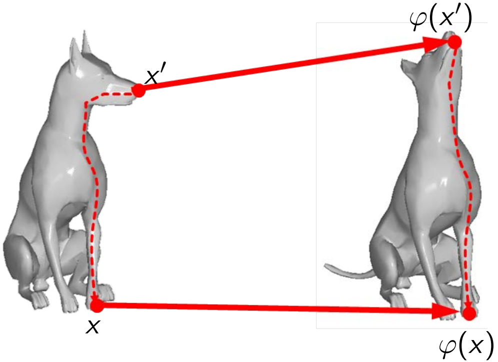

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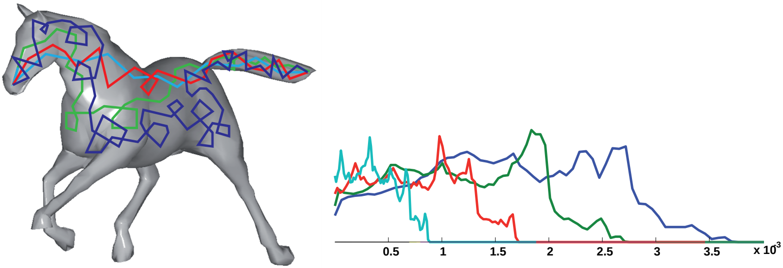
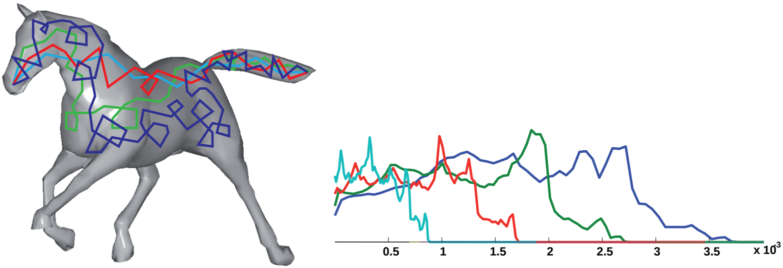
- Solved using *generalized multidimensional scaling*

Diffusion distance distributions



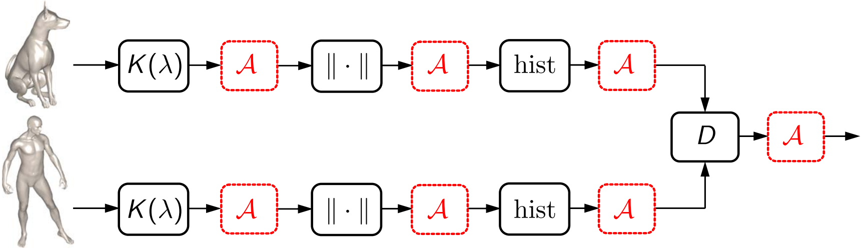
- Represent shape as *distribution* of diffusion distances

Diffusion distance distributions



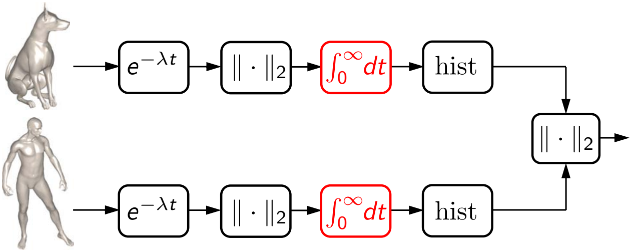
- Represent shape as *distribution* of diffusion distances
- Compare shapes using *divergence* of distributions

Diffusion distance distributions



Diffusion distance distributions

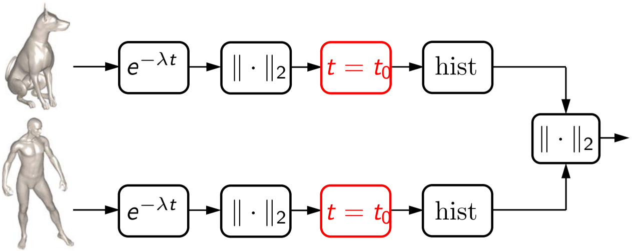
Particular case I: Rustamov's GPS embedding



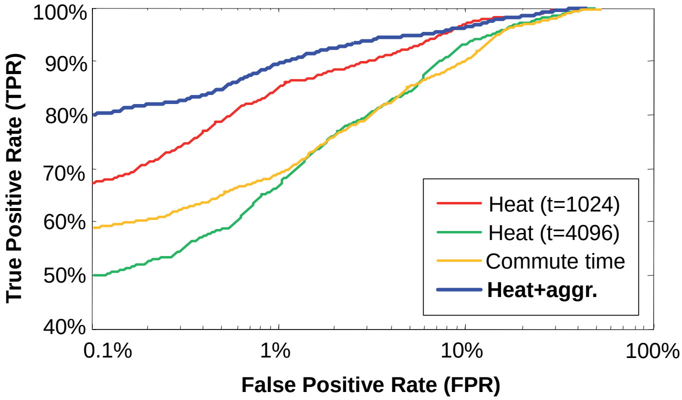
$$d^2_{\mathrm{CT}}(x,y) = \int_0^\infty \underbrace{\sum_{i \geq 0} e^{-2\lambda_i t} (\phi_i(x) - \phi_i(y))^2}_{\text{Diffusion distance } d_t^2(x,y)} dt$$

Diffusion distance distributions

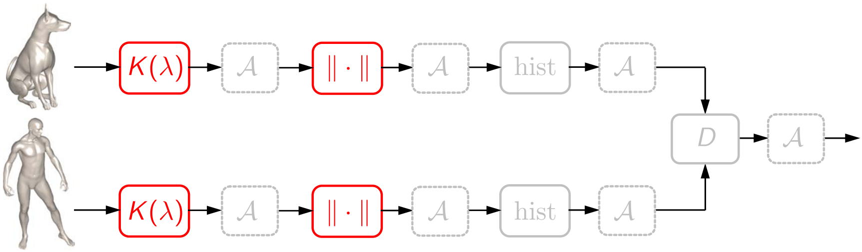
Particular case II: Mahmoudi & Sapiro



Diffusion distance distributions



Generalizations



- Diffusion distances generated by other norms, e.g. $\|\cdot\|_{L^1(X)}$

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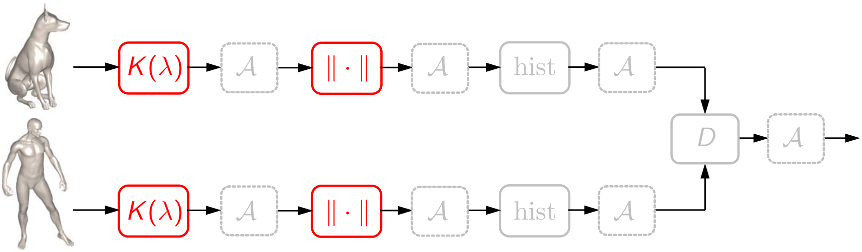
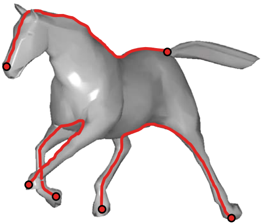
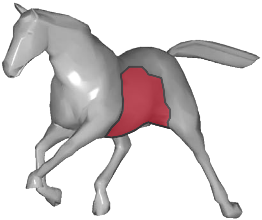


- Diffusion distances generated by other norms, e.g. $\|\cdot\|_{L^1(X)}$
- Construct (or learn) *optimal* task-specific diffusion kernels

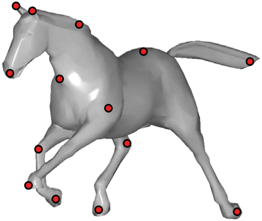
Local structure



Global structure
Metric space



Glocal structure
Stable regions



Local structure
Point descriptors

Diffusion kernel descriptors

- Associate each point x with a vector $(k_{t_1}(x,x), \ldots, k_{t_n}(x,x))$

(Sun *et al.*, SGP'09)

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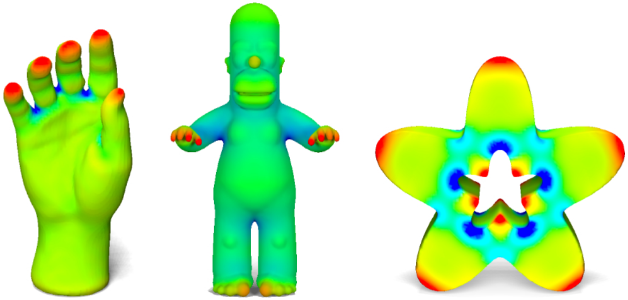
Diffusion kernel descriptors

- Associate each point x with a vector $(k_{t_1}(x,x), \dots, k_{t_n}(x,x))$
- Multi-scale *point-wise descriptor*
- *Heat kernel signature* (HKS): $x \mapsto (h_{t_1}(x,x), \dots, h_{t_n}(x,x))$

(Sun *et al.*, SGP'09)

Heat kernel signature



$$h_t(x,x) = \frac{1}{4\pi t}\left(1 + \frac{1}{3}K(x)t + \mathcal{O}(t^2)\right)$$

$K(x)$ = Gaussian curvature at point x

(Sun *et al.*, SGP'09)

Scale invariance

- **Original shape**
- Eigenvalues λ_i
- Eigenfunctions $\phi_i(x)$
- Heat kernel $h_t(x,x)$

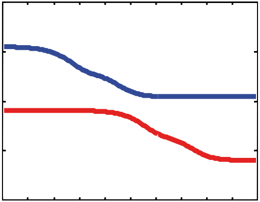
- **Scaled by $\frac{1}{\alpha}$**
- Eigenvalues $\alpha^2\lambda_i$
- Eigenfunctions $\alpha\phi_i(x)$
- Heat kernel $\alpha^2 h_{\alpha^2 t}(x,x)$
- **Not scale invariant!**

Scale invariance

Log scale space

$h_{e^\tau} \to \alpha^2 h_{e^{\tau+\beta}}$

$\beta = \log \alpha^2$



Scale $\to$ shift + factor

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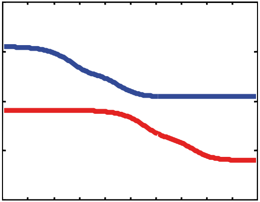
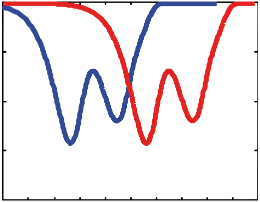


Scale $\to$ shift + factor

Log + derivative

$$\frac{d}{d\tau}\left(\log \alpha^2 + \log h_{e^{\tau+\beta}}\right)$$

$$= \frac{d}{d\tau} \log h_{e^{\tau+\beta}}$$



Undo factor

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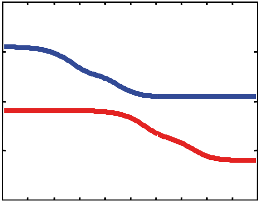
Log + derivative

$$\frac{d}{d\tau}\left(\log \alpha^2 + \log h_{e^{\tau+\beta}}\right) = \frac{d}{d\tau}\log h_{e^{\tau+\beta}}$$

Fourier magnitude

$$\mathcal{F}\left\{\frac{d}{d\tau}\log h_{e^{\tau+\beta}}\right\} = e^{\beta i \omega \pi}\mathcal{F}\left\{\frac{d}{d\tau}\log h_{e^{\tau}}\right\}$$



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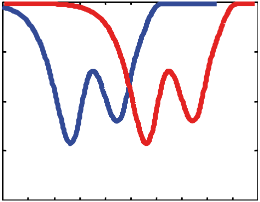

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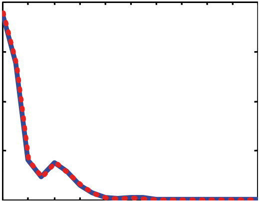

Undo shift

Scale invariant heat kernel signature

(B&Kokkinos, CVPR'09)

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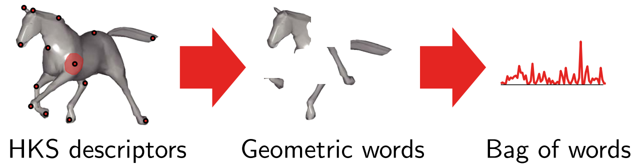
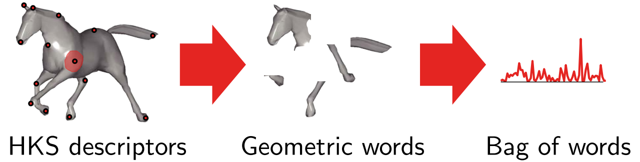
ShapeGoogle



- Given point-wise descriptor $\mathbf{h}(x)$

ShapeGoogle



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- *Quantize* each $\mathbf{h}$ in a fixed vocabulary $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$

ShapeGoogle

HKS descriptors Geometric words Bag of words

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$$\mathbf{H} = \int_X \mathbf{v}(x) da(x)$$

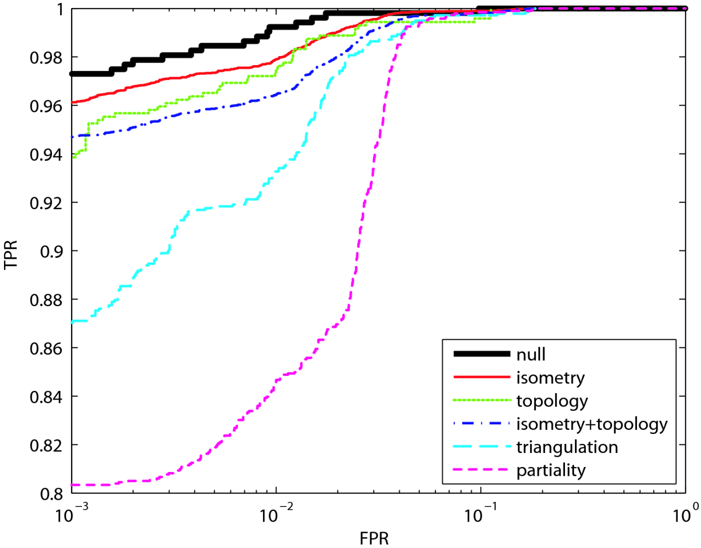
ShapeGoogle

HKS descriptors | Geometric expressions | Spatially-sensitive bag of words

- *Spatially-sensitive* bags of pairs of words

$$\mathbf{H} = \int_{X \times X} \mathbf{v}(x)\mathbf{v}(y)h_t(x,y)da(x)da(y)$$

ShapeGoogle



Partial matching

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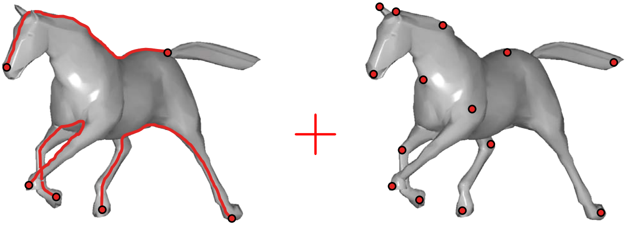
- *Ambrosio-Tortorelli* approximation

$$\min_{u,\rho} \left\| \int_{X} \mathbf{v} u da - \mathbf{H}' \right\|^2 + \mu_1 \int_X \rho^2 \|\nabla u\|^2 da + \mu_2 \epsilon \int_X \|\nabla \rho\|^2 da$$

$$+ \frac{\mu_2}{4\epsilon} \int_X (1-\rho)^2 da \ \ \text{s.t.} \ \int_X u da = A(Y')$$

Partial matching

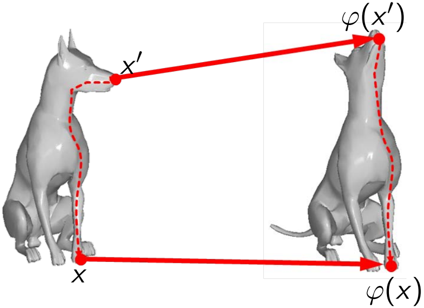
Global + local structures



Global structure
Metric space

Local structure
Point descriptors

Correspondence *encore*



- Find *minimum distortion* correspondence

$$\min_{\varphi:X\to Y} \|d_X - d_Y \circ (\varphi \times \varphi)\|$$

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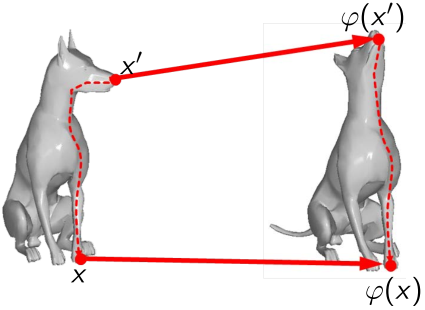


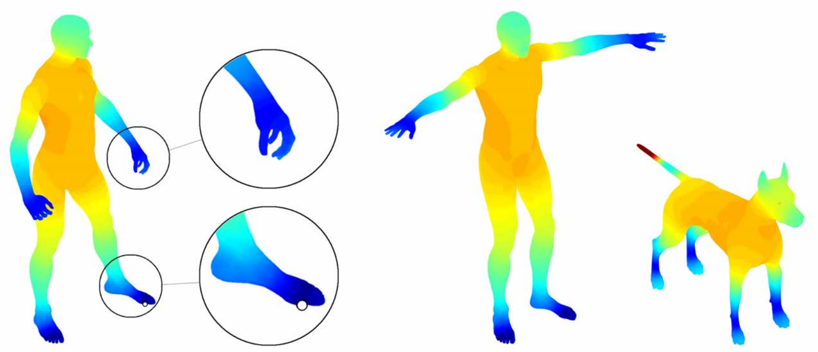
- Find *minimum distortion* correspondence

$$\min_{\varphi:X\to Y} \|d_X - d_Y \circ (\varphi \times \varphi)\| + \mu \|\mathbf{h}_X - \mathbf{h}_Y \circ \varphi\|$$

d_X, d_Y – global structures
$\mathbf{h}_X, \mathbf{h}_Y$ – local structures

- Combine local and global structure distortion

Heat kernel signature *encore*



- Poor spatial feature localization!

Aubry *et al.*, CVPR'11; B, PAMI'11

Wave kernel signature

- Different physical model: a quantum particle with initial energy distribution $f(\nu)$.

Aubry *et al.*, CVPR'11

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- Solution in spectral domain

$$\psi(x,t) = \sum_{k\geq 0} e^{i\nu_k t} f(\nu_k)\phi_k(x)$$

Aubry *et al.*, CVPR'11

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$$f_e(\nu) \propto \exp\left(-\frac{(\log e - \log \nu)^2}{2\sigma^2}\right)$$

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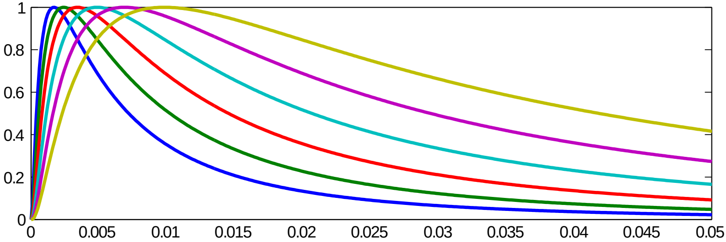
- Each point is associated the *wave kernel signature*

$$\mathbf{p}(x) : x \quad \mapsto \quad (p_{e_1}(x), \ldots, p_{e_n}(x))$$

Aubry *et al.*, CVPR'11

Wave kernel signature



- Collection of *band pass filters*

$$\mathbf{p}(x) = \sum_{k\geq 0} \begin{pmatrix} p_1(\nu_k) \\ \vdots \\ p_n(\nu_k) \end{pmatrix} \phi_k^2(x)$$

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Aubry *et al.*, CVPR'11

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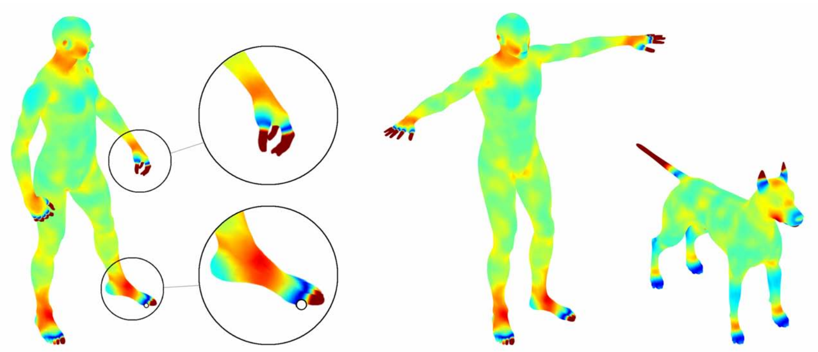
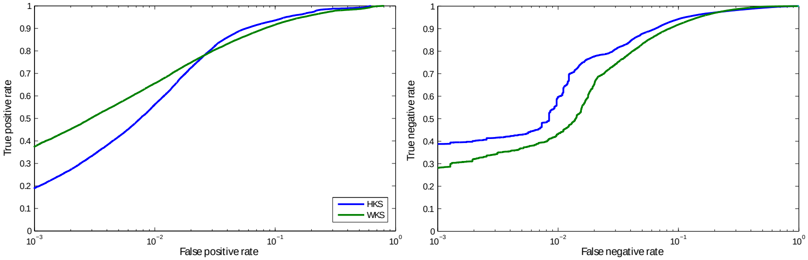


- Better spatial feature localization
- Lower discriminativity

Aubry *et al.*, CVPR'11; B, PAMI'11

Waves vs Heat



- WKS has higher *sensitivity* than HKS (better at low FPR)

B, PAMI'11

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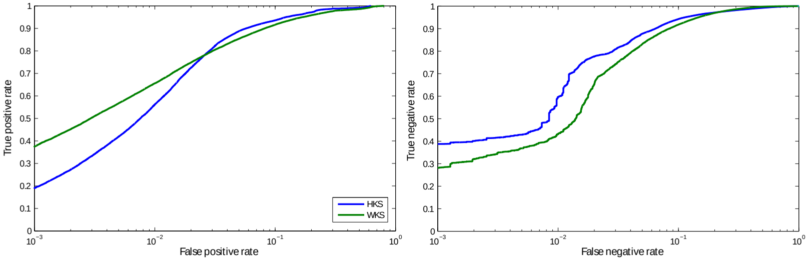

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B, PAMI'11

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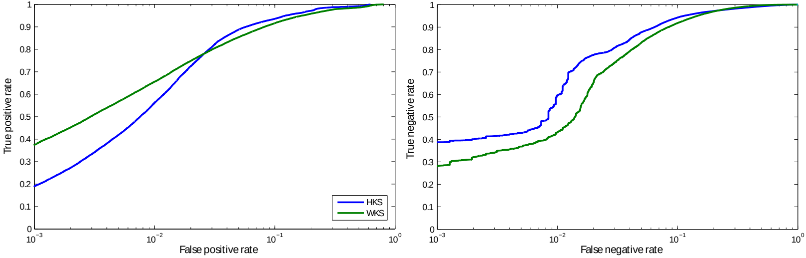

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B, PAMI'11

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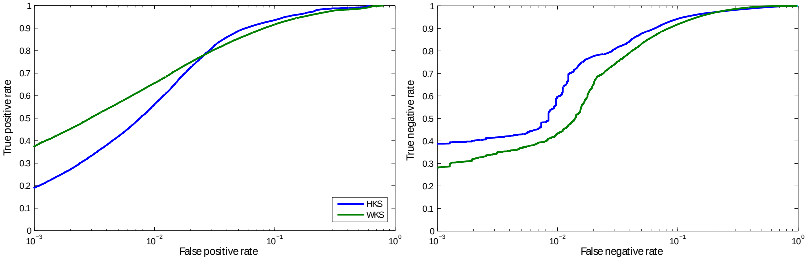

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- HKS is better for shape *retrieval* problems

B, PAMI'11

An ideal descriptor

- **Localization**: changes a lot under a small displacement on the surface.

B, PAMI'11

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B, PAMI'11

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B, PAMI'11

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B, PAMI'11

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- **Efficiency**: maximum information per number of dimensions.

B, PAMI'11

Optimal spectral descriptors

- Build *optimal spectral descriptor* of the form

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B, PAMI'11

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B, PAMI'11

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B, PAMI'11

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B, PAMI'11

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  - pass frequencies with large signal content (discriminative geometric features)
- Hard to model axiomatically
- ...yet easy to *learn* from examples!

B, PAMI'11

Parametrization

- To do learning, we need a *finite* set of parameters

B, PAMI'11

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B, PAMI'11

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B, PAMI'11

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- Represent responses as

$$\begin{pmatrix} f_1(\nu) \\ \vdots \\ f_n(\nu) \end{pmatrix} = \mathbf{A} \begin{pmatrix} b_1(\nu) \\ \vdots \\ b_m(\nu) \end{pmatrix}$$

with the matrix of parameters $\mathbf{A}$

B, PAMI'11

Parametrization

- Select sufficiently large s for which $\nu_s \geq \nu_{\max}$

B, PAMI'11

Parametrization

- Select sufficiently large s for which $\nu_s \geq \nu_{\max}$
- Represent the point descriptor

$$
\begin{aligned}
\mathbf{p}(x) &= \sum_{k\geq 0} \begin{pmatrix} f_1(\nu_k) \\ \vdots \\ f_n(\nu_k) \end{pmatrix} \phi_k^2(x) \\
&\approx \mathbf{A} \begin{pmatrix} b_1(\nu_1) & \cdots & b_1(\nu_s) \\ \vdots & \ddots & \vdots \\ b_m(\nu_1) & \cdots & b_m(\nu_s) \end{pmatrix} \begin{pmatrix} \phi_1^2(x) \\ \vdots \\ \phi_s^2(x) \end{pmatrix} = \mathbf{Ag}(x)
\end{aligned}
$$

B, PAMI'11

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- *Geometry vector* **g** consistently represents all geometric information at point x.

B, PAMI'11

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- x a point, x_+ a knowingly similar point (*positive*), x_- a knowingly dissimilar point (*negative*).

B, PAMI'11

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B, PAMI'11

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$$\begin{aligned} d_\pm^2 &= \|\mathbf{p} - \mathbf{p}_\pm\|^2 = \|\mathbf{A}(\mathbf{g} - \mathbf{g}_\pm)\|^2 \\ &= (\mathbf{g} - \mathbf{g}_\pm)^\mathrm{T} \mathbf{A}^\mathrm{T} \mathbf{A} (\mathbf{g} - \mathbf{g}_\pm). \end{aligned}$$

B, PAMI'11

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- *Mahalanobis distance* on geometry vectors space.

B, PAMI'11

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- Simultaneously *maximize* $d(\mathbf{p}, \mathbf{p}_-)$ and *minimize* $d(\mathbf{p}, \mathbf{p}_+)$.
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$$\begin{aligned} d_\pm^2 &= \|\mathbf{p} - \mathbf{p}_\pm\|^2 = \|\mathbf{A}(\mathbf{g} - \mathbf{g}_\pm)\|^2 \\ &= (\mathbf{g} - \mathbf{g}_\pm)^{\mathrm{T}} \mathbf{A}^{\mathrm{T}} \mathbf{A} (\mathbf{g} - \mathbf{g}_\pm). \end{aligned}$$

- *Mahalanobis distance* on geometry vectors space.
- Mahalanobis *metric learning* problem.

B, PAMI'11

Learning

- Taking expectation over positive/negative pairs

$$\begin{aligned}\mathbb{E}\{d_\pm^2\} &= \mathbb{E}\left\{\|\mathbf{p}-\mathbf{p}_\pm\|^2\right\} = \mathbb{E}\left\{(\mathbf{g}-\mathbf{g}_\pm)^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}(\mathbf{g}-\mathbf{g}_\pm)\right\} \\ &= \mathrm{tr}\left\{\mathbf{A}\mathbb{E}((\mathbf{g}-\mathbf{g}_\pm)(\mathbf{g}-\mathbf{g}_\pm)^{\mathrm{T}})\mathbf{A}^{\mathrm{T}}\right\} \\ &= \mathrm{tr}\left\{\mathbf{A}\mathbf{C}_\pm\mathbf{A}^{\mathrm{T}}\right\}\end{aligned}$$

Strecha,Fua,BB, PAMI'11; B, PAMI'11

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$$\min\ (1-\alpha)\mathbb{E}(d_{+}^2) - \alpha\mathbb{E}(d_{-}^2)$$

Strecha,Fua,BB, PAMI'11; B, PAMI'11

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Strecha,Fua,BB, PAMI'11; B, PAMI'11

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- α controls tradeoff between *sensitivity* ($\alpha \to 1$) and *specificity* ($\alpha \to 0$).

Strecha,Fua,BB, PAMI'11; B, PAMI'11

Learning

- Scale of **A** is arbitrary!

B, PAMI'11

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- Scale of $\mathbf{A}$ is arbitrary!
- Recall *efficiency*: dimensions of $\mathbf{p}$ should be *statistically independent*

B, PAMI'11

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B, PAMI'11

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$$\mathbf{I} = \mathbb{E}(\mathbf{p}\mathbf{p}^{\mathrm{T}}) = \mathbf{A}\mathbb{E}(\mathbf{g}\mathbf{g}^{\mathrm{T}})\mathbf{A}^{\mathrm{T}} = \mathbf{A}\mathbf{C}\mathbf{A}^{\mathrm{T}}$$

B, PAMI'11

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B, PAMI'11

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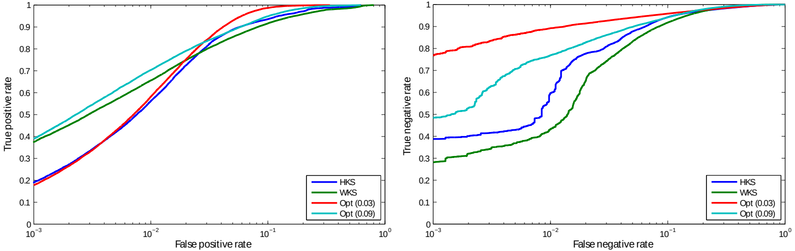
- Solution: $\mathbf{A} = \mathbf{U}_n^{\mathrm{T}}\mathbf{C}^{-1/2}$ where $\mathbf{U}_n$ are the n smallest eigenvectors of $\mathbf{C}^{-1/2}\mathbf{D}_{\alpha}\mathbf{C}^{-1/2}$.

B, PAMI'11

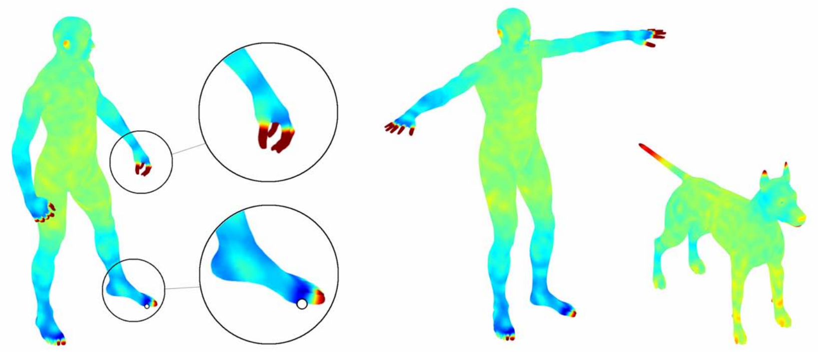
Optimal spectral descriptor



- Better specificity and sensitivity than HKS and WKS

B, PAMI'11

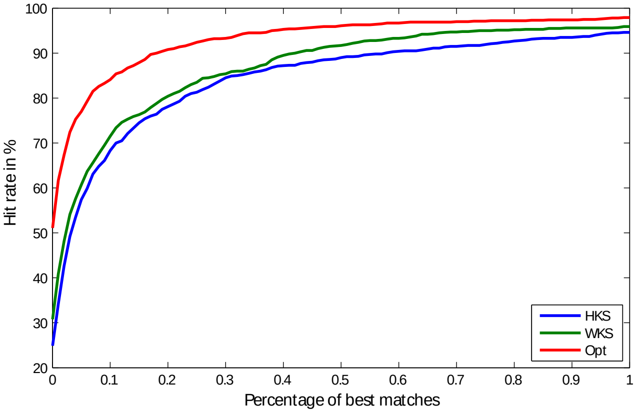
Optimal spectral descriptor



- Better spatial feature localization than WKS
- Better discriminativity than HKS

B, PAMI'11

Optimal spectral descriptor

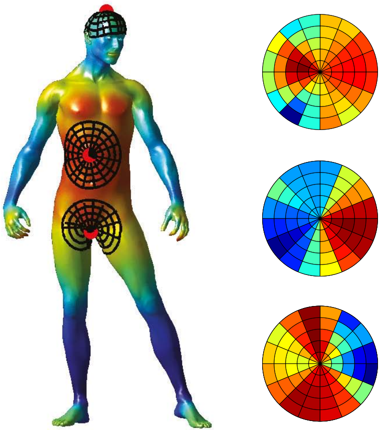


- Better performance in correspondence problems

B, PAMI'11

Intrinsic shape contexts

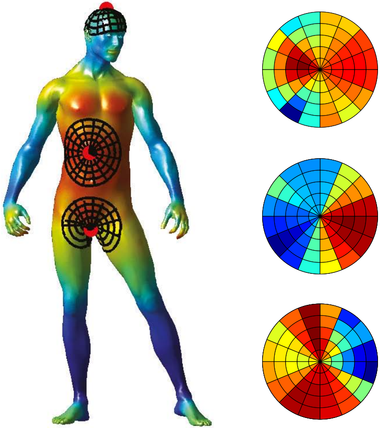
- Spectral descriptors lack *orientation* information



Kokkinos,Litman,BB, CVPR'12

Intrinsic shape contexts

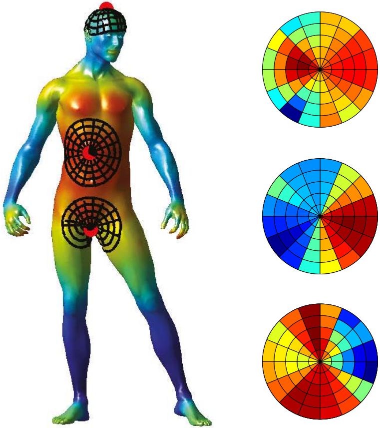
- Spectral descriptors lack *orientation* information
- Given a vector field **p** on surface, compute its distribution over a local polar system of coordinate



Kokkinos,Litman,BB, CVPR'12

Intrinsic shape contexts

- Spectral descriptors lack *orientation* information
- Given a vector field **p** on surface, compute its distribution over a local polar system of coordinate
- *Intrinsic shape context* (ISC) – a meta-descriptor



Kokkinos,Litman,BB, CVPR'12

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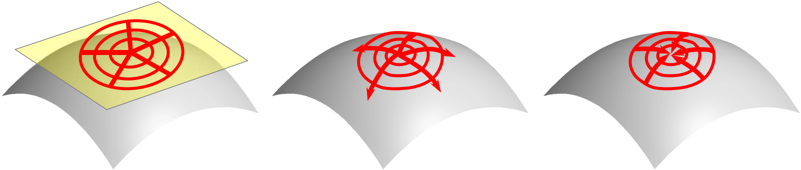
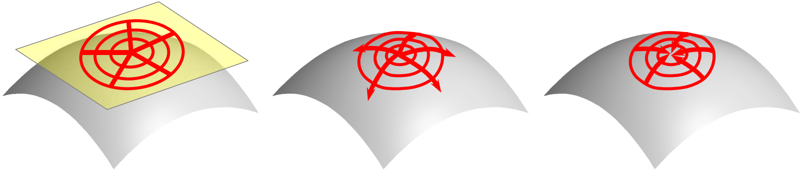


Tangent plane map   Inward shooting   Outward shooting

- Problem I: no global coordinate system

Kokkinos,Litman,BB, CVPR'12

Intrinsic shape contexts



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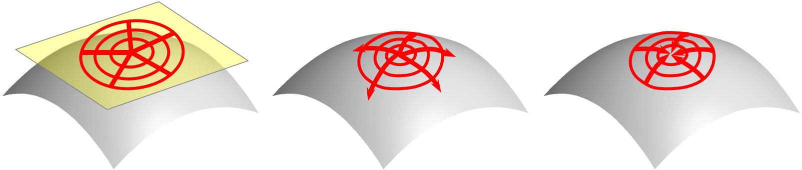

Tangent plane map | Inward shooting | Outward shooting

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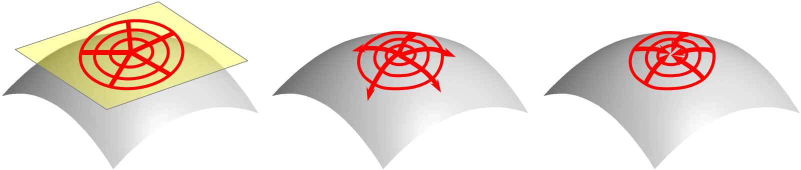

Tangent plane map Inward shooting Outward shooting

- Problem I: no global coordinate system
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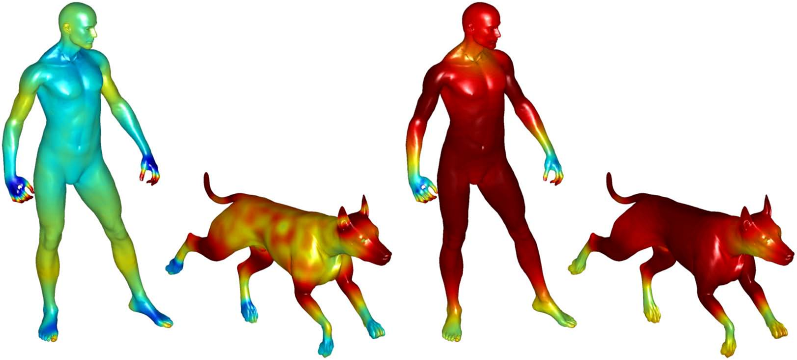
- Problem II: arbitrary angular coordinate
- Undone using Fourier transform modulus

Kokkinos,Litman,BB, CVPR'12

Intrinsic shape contexts

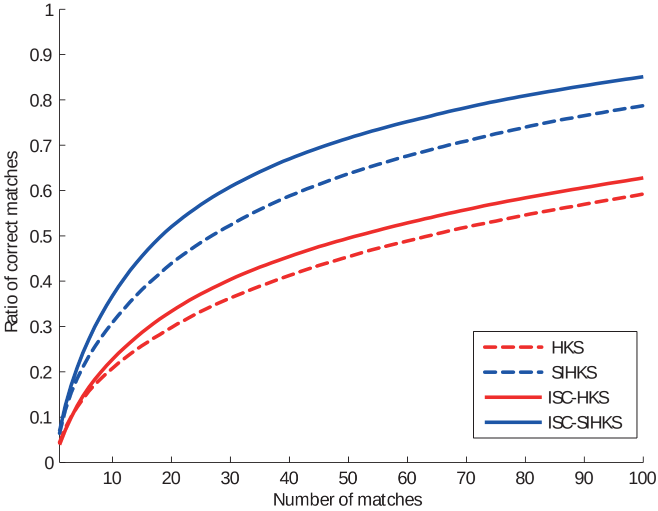


Scale Invariant HKS

Intrinsic Shape Context

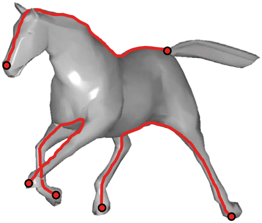
Kokkinos,Litman,BB, CVPR'12

Intrinsic shape contexts

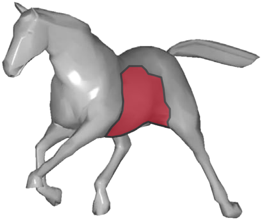


Kokkinos,Litman,BB, CVPR'12

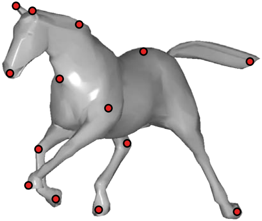
Glocal structure



Global structure
Metric space



Glocal structure
Stable regions



Local structure
Point descriptors

Component trees

- Measure proximity $d(x, y)$ in some local neighborhood $y \in \mathcal{N}(x)$

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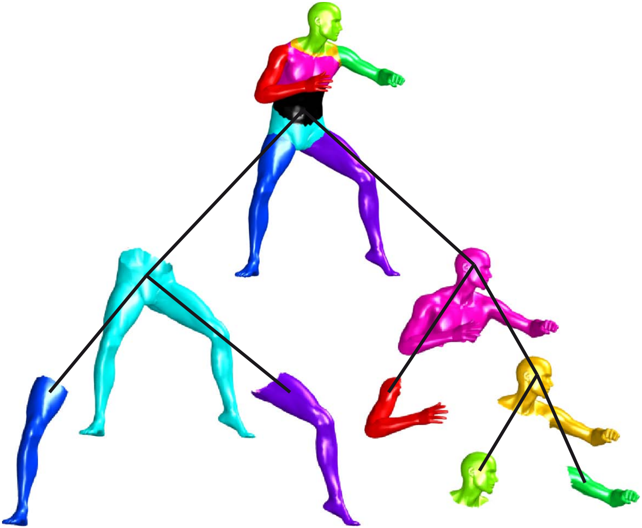
- Hierarchy described as *component tree*

Component trees



Maximally stable components

- *Stability* of a component

$$\sigma(C_t) = \frac{A(C_t)}{\frac{dA(C_t)}{dt}}$$

- Measures relative change of area as function of change of threshold
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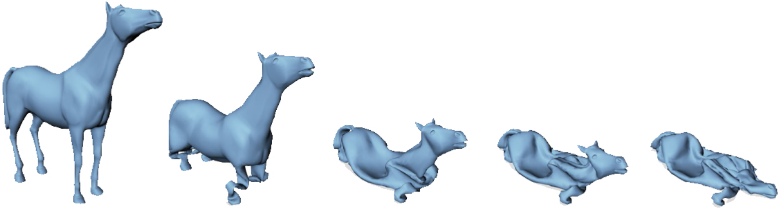
Maximally stable components

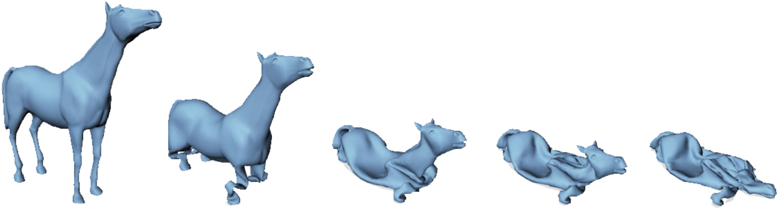
Volumetric diffusion geometry



- *Boundary isometry* does not always represent a realistic deformation

(image: Sumner)

Volumetric diffusion geometry



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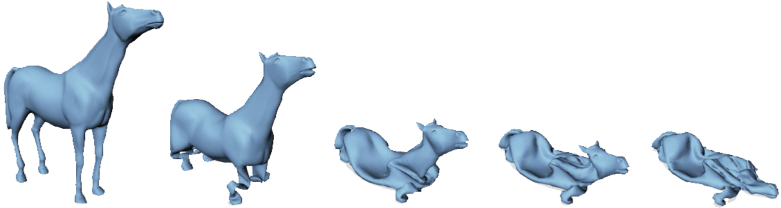

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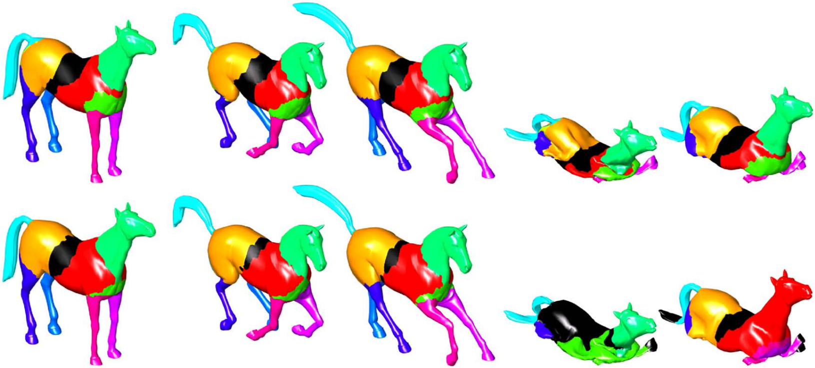
- Solution: *volumetric* diffusion geometry

(image: Sumner)

Volumetric diffusion geometry

Volumetric maximally stable components



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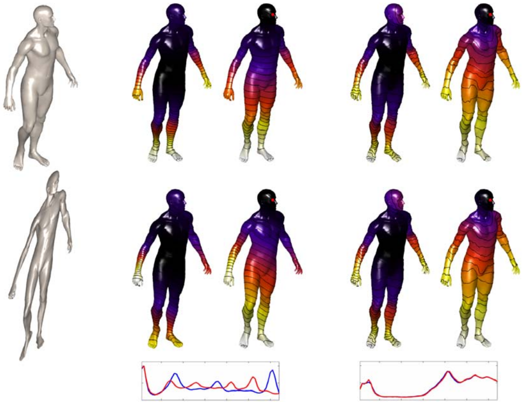
- Positive definite only on *convex* surfaces
- Construct g from $\hat{g}$ enforcing positivity of eigenvalues
- g is a *valid metric* on manifolds with non-vanishing curvature

Affine invariance

- Define *equi-affine Laplacian* Δ_g
- **Equi-affine Laplacian + scale-invariance = affine-invariance**



Conclusion

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- Extensions

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