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Stable Morse Decompositions for Piecewise Constant Vector Fields on Surfaces: Supplementary Material

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Appendix A: Admissibility of the flow in degenerate cases

Admissibility is a technical condition that ensures that the fixed point index of the flow is well-defined [Gor06]. Recall that a flow is admissible if and only if it is upper semicontinuous (i.e. if the limit of a convergent sequence of trajectories is a trajectory) and if there exists $h > 0$ such that, for any point x, the set of trajectory segments $S(x,h)$ starting at x and defined on the time interval $[0,h]$ is contractible.

The proof of admissibility of the flow induced by a PC vector field in [SZ10] is restricted to the non-degenerate case, i.e. it assumes that: (i) $f(\Delta)$ is not parallel to any of the Δ's edges (in particular, it is nonzero) for any triangle Δ, (ii) $f(e)$ is not the zero vector. In the degenerate cases, trajectories are defined as curves obtained by concatenating feasible segments described in Section 4, for $F(\Delta) = \{f(\Delta)\}$, *without* using any simulation of simplicity mentioned in that section. In Section 6 we argue that the fixed point index of a Morse set does not depend on the choice of a feasible PC vector field. The argument is based on a linear homotopy f_h connecting two feasible PC vector fields f and f_0. The problem is that one cannot guarantee that the vector fields f_h are non-degenerate for all values $h \in [0,1]$ (even if f and f_0 are non-degenerate), and therefore admissibility of the flow must be established also for degenerate cases to ensure the validity of the argument.

Fortunately, extending the result of [SZ10] to the degenerate case is quite straightforward. The only cases that requires special treatment involve an edge e such that the vector $f(\Delta)$ is parallel to e in each of its incident triangles Δ (we call such edge *special*). Then, $F(e)$ does not consist of a single vector according to the definition in Section 4.2. Simple trajectory segments moving along such an edge e can be defined as continuous functions $\sigma : [t_1,t_2] \to e$ such that for each $s,t \in [t_1,t_2]$ such that $s \neq t$, $(\sigma(s)-\sigma(t))/(s-t) \in F(e)$ (this is inspired by the theory of differential inclusions [Gor06]).

With this interpretation, the proofs of admissibility of the flow in [SZ10] easily carry over to the degenerate case. The proof of upper semicontinuity follows the same lines. Special edges are easy to deal with since the flow in both triangles incident to such an edge is parallel to it. Therefore, trajectories exit or such edges only at the endpoints.

The crux of the rest of the proof in [SZ10] is to argue that the set $S(x,h)$ of all trajectory segments out of a stationary vertex x defined on $[0,h]$ is acyclic (i.e. has trivial homology in all dimensions) if the union of all trajectories in $S(x,h)$ is contained in the star of a mesh vertex v (i.e. the interior of the union of all mesh triangles incident upon v). The methods developed in [SZ10] carry over to all degenerate cases except for the case of x belonging to a special but non-critical edge e incident upon v. We focus only on the new components of the proof below.

First, assume that $x = v$. If v is stationary, the technique of [SZ10] directly applies (in fact, it also applies if x is any stationary point, not necessarily a vertex). Therefore, we assume v is not stationary. In this case, the proof technique in [SZ10] uses induction with respect to unstable directions of x. Following their notation, let d_i be the unstable direction of x, ordered counterclockwise, $i = 1,2,\ldots,n$ and S_i consists of all trajectories in $S := S(v,h)$ between directions d_i and d_n. Their proof constructs a deformation retraction of S_i to S_{i+1}. The construction works verbatim in the degenerate setting, except for a few cases that require special treatment.

The first case is that of d_i being a special edge. Then, trajectories between d_i and d_{i+1} move toward d_i as seen from v (Figure 1). If this is the case, the trick is to first deform trajectories that follow d_i (let F be the set of all such trajectories) to equalize their velocity to make it equal to $f(\Delta)$, where Δ is the triangle incident upon the edge leaving v in the direction d_i on the counterclockwise side. For any $\sigma \in F$ we set $\sigma_t(s) = (1-t)\sigma(s) + t(v + sf(\Delta))$ and define the deformation by $H(\sigma,t) = \sigma_t$ if $\sigma \in F$ and $H(\sigma,t) = \sigma$ otherwise.

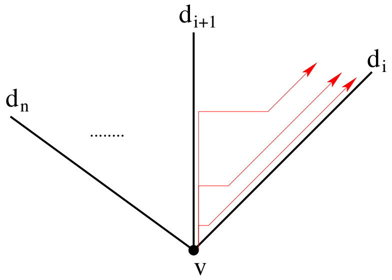

Figure 1: *First case: d_i special. Deform the trajectories following d_i to make their velocity equal to the velocity of trajectories on the counterclockwise side of d_i. This reduces this setting to one discussed in [SZ10].*

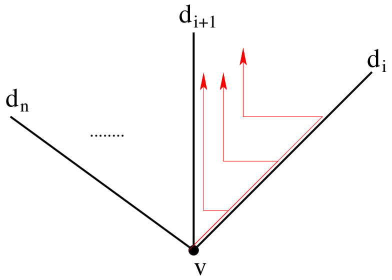

Figure 2: *Second case: d_{i+1} pointing along a special edge. σ_u is the (unique) trajectory that leaves d_i at time u. σ_0 follows d_{i+1} with velocity equal to the velocity of the trajectories just to the right of d_{i+1}. Roughly, the deformation works by gradually decreasing u to zero for such trajectories to deform S_i to S_{i+1}.*

Intuitively, H gradually increases or decreases the velocity of trajectories in F to $f(\Delta)$. After that, one can use the deformation in [SZ10], pretending that trajectories move along d_{i+1} only with velocity $f(\Delta)$. It is not hard to see that d_{i+1} cannot point along a special edge since d_i is assumed to.

The second case is that of d_{i+1} being a special edge. In this case, trajectories move from d_i to d_{i+1} as seen from v and d_i is not special (Figure 2). One can parametrize trajectories leaving d_i using time at which they exit d_i. Thus, for $u > 0$, σ_u leaves d_i at time u and then moves clockwise around v toward d_{i+1}. Note that it never reaches d_{i+1} since it extends along a special edge e. By σ_0 we denote the trajectory moving along e with constant velocity $f(\Delta)$, where Δ is the triangle incident upon e on clockwise side as seen from v. The deformation of S_i to S_{i+1} can be defined using the same formula as in [SZ10]: $H(\sigma_t, s) = \sigma_{(1-s)t}$, and $H(\sigma, s) = \sigma$ for all $\sigma \notin \{\sigma_t | t \in [0,h]\}$.

Finally, in contrast to the PC case, S_n does not consist of exactly one trajectory, if d_n follows a special edge. But then, one can linearly deform trajectories in S_n to a single one moving with a fixed velocity, in the same way as described in the paragraph discussing the first case above.

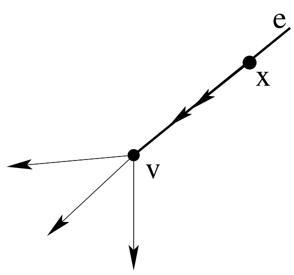

Figure 3: *Last new case: x on a special edge e incident to v. Trajectories starting at x follow e until they reach v. Their velocity vector may vary since $F(e)$, in the generic case, consists of an interval of vectors pointing along e, in the direction from x to v. Nothing is assumed about the vector field near v: in particular, trajectories reaching v may possibly leave along one of its multiple unstable directions.*

It remains to deal with the case of x on a special (but non-critical) edge e incident upon v. Then, trajectories can only leave the interior of e through v (Figure 3). In this case, one can gradually slow down the trajectories when they travel along e to the minimum velocity $\vec{u}$ in $F(e)$, and follow them past v until time h. By doing that, one obtains a deformation of S into the set of trajectories $S' \subset S$ that move along e with constant velocity $\vec{u}$, homeomorphic to $S(v, h-T)$ (the homeomorphism is simply a restriction to $[T,h]$ followed by a shift to the left by T), where $T = \frac{|v-x|}{|\vec{u}|}$ is the time needed for a trajectory in S moving along e with velocity $\vec{u}$ to reach v. Note that $S(v, h-T)$ has already been proven to be acyclic.

Technically, the deformation of S to S' can be obtained using the following scheme, in which $\tau(\sigma)$ is the first time $u \in [0,h]$ such that $\sigma(u) = v$ for $\sigma \in S$. Note that τ is a continuous function on S. Let σ_t, for $t \in [0,1]$, be the trajectory that:

(i) follows σ for all times between 0 and $t_1 := \max(\tau(\sigma) - Tt, 0)$, i.e. $\sigma_t(s) = \sigma(s)$ for $s \in [0, t_1]$,
(ii) moves along e, from $\sigma(t_1)$ to v, with velocity $\vec{u}$ until it reaches v, i.e. for times $s \in [t_1, t_2]$, where $t_2 = t_1 + \frac{|v-\sigma(t_1)|}{|\vec{u}|}$, $\sigma_t(s) = \sigma(t_1) + (s - t_1)\vec{u}$.
(iii) Follows σ past time t_2, i.e. $\sigma_t(s) = \sigma(s - t_2 + \tau(\sigma))$.

With the above definition, the mapping $H(\sigma, t) = \sigma_t$ defines a deformation of S to S'.

Appendix B: Definition of Morse sets

The definition and proof of correctness of Morse decomposition [SZ10] applies to the setting of this paper with minor changes. A Morse set defined by a strongly connected component A can be defined as the union of (i) all critical triangles adjacent to A (note that critical triangles do not exist

Dataset	Stability	Figure(s)	Refinement iterations	Maximum graph size		Running time [s]	Peak memory usage [GB]
				Nodes[Millions]	Arcs [Milions]		
Gas engine	4%	8	8	1.00	64.6	211	4.55
	10%	9	7	1.67	78.6	247	5.63
	11%	9	7	1.84	85.2	266	6.11
Diesel engine	2.5%	10	8	0.96	60.6	201	4.42
	2.6%	10	8	1.05	70.6	237	5.12
Cooling jacket	0.1%	11,12	10	1.63	28.9	170	2.43
	0.25%	12	9	1.92	42.7	195	3.42
	0.5%	11,12	8	2.15	50.2	210	3.98
	1%	11,12	7	2.05	46.4	183	3.69
	2%	11	6	2.16	37.6	150	3.13
	2.5%	12	6	2.55	46.7	179	3.82
	5%	11	5	2.45	30.2	118	2.68

Table 1: *Runtime statistics for stable Morse decompositions*

Dataset	Figure	Refinement iterations	Maximum graph size		Running time [s]	Peak memory usage [GB]
			Nodes [Millions]	Arcs [Millions]		
Gas engine	14	8	0.73	64.4	203	4.47
Gas engine (1 subdivision)	14	8	0.98	74.0	257	5.22
Gas engine (2 subdivisions)	14	8	1.86	80.9	336	6.15
Gas engine (3 subdivisions)	14	7	2.83	35.2	369	4.30
Diesel engine	15	9	0.54	49.5	188	3.59
Hurricane Isabel slice	15	7	0.64	39.0	300	4.25

Table 2: *Runtime statistics for Envelopes*

in the plain PC setting) and (ii) middle segments of trajectories of feasible vector fields captured by paths of length $2D+2$ in A, where D is the maximum degree of a vertex. In the degenerate cases, simple trajectory segments are defined as segments of constant velocity contained in a single mesh triangle *or* as segments described in Appendix A moving along special edges. With the above adjustments, the proof in [SZ10] extends to the setting of this paper.

Appendix C: Morse sets of Lipschitz continuous vector fields

In this section, we outline a proof showing that if g is a Lipschitz continuous vector field defined on a triangulated planar domain D and F is a CVPC vector field such that $g(\Delta) \subset F(\Delta)$ for every triangle Δ, then any strongly connected component A of a super-transition graph $\mathcal{G}$ of F defines a Morse set of g. Moreover, the fixed point index of M_A with respect to g is the same as the fixed point index of the Morse set of F.

A proof could be obtained by following the technique of [SZ10]. We include a different argument below.

The Morse sets can be defined as follows. Let M_A, for a strongly connected component A of $\mathcal{G}$, be the union of (i) all critical triangles incident to A and (ii) all trajectories of g contained in a single triangle that start and end in an n-set in A. Note that M_A and M_B may not be disjoint if A and B are different strongly connected components of $\mathcal{G}$. However, define $\hat{M}_A$ as the invariant part of M_A, i.e. the union of all trajectories of g contained in M_A. Then, $\hat{M}_A$ and $\hat{M}_B$ are disjoint for any two different strongly connected components A and B of $\mathcal{G}$. Otherwise, a trajectory contained both in M_A and M_B would exist. The trajectory would intersect edges only at points that belong both to an n-set in A and to an n-set in B. There would be at least two such intersections (since critical triangles contain all stationary points of g and no one of them is incident both to an n-set in A and to an n-set in B). Assume one of the intersection points is in n-sets a_1 and b_1 in A and B (respectively) and a later one – in $a_2 \in A$ and $b_2 \in B$. Since there is a trajectory segment connecting a_1 and b_2, there is a path from a_1 to b_2 in $\mathcal{G}$. Similarly, since there is a trajectory segment connecting b_1 and a_2 – there is a path from b_1 to a_2. We conclude that A and B are in fact the same strongly connected component, which is a contradiction.

To show that the sets $\hat{M}_A$ defined above are indeed valid Morse sets, take any trajectory $\sigma : (-\infty,\infty) \rightarrow D$ of g through a point x_0 not contained in any $\hat{M}_A$. By the property proved in Section 7.2, that trajectory is represented by a bidirectional path π in $\mathcal{G}$. Note that π extends indefinitely forward (backward), unless the trajectory is converging to a stationary point when followed forward (respectively, backward). π is not contained in a single strongly connected component of $\mathcal{G}$. Therefore, it has to originate in a strongly connected component A and terminate in a different strongly connected component B. This means that σ stays in M_A for sufficiently large negative times and in M_B for sufficiently large positive times. We conclude that σ converges to $\hat{M}_A$ and $\hat{M}_B$ when followed backward (respectively, forward) in time.

The fixed point index of a Morse set $\hat{M}_A$ of a feasible PC vector field f defined by A is the same as the index of the respective Morse set M_A of g. To see why, define an upper semicontinuous convex valued vector field g_t for $t \in [0,1]$ by

$$g_t(x) := Hull\left\{(1-t)g(x) + tf(\Delta)|x \in \Delta\right\},$$

where Δ denotes a mesh triangle. Note that $g_t(x)$ can contain the zero vector only at mesh vertices and points in critical triangles or critical edges. The set of fixed points in n-sets of A or in critical triangles incident to A stays isolated for each g_t. By the homotopy invariance of the fixed point index for multivalued maps [Gor06], the index on a Morse set is the same for $g = g_0$ and g_1. The solutions of the differential inclusion $\dot{x} \in g_1(x)$ are identical to trajectories of f except that, for some non-stationary vertices v, $g_1(x)$ may contain the zero vector. In other words, the multivalued flow defined by the differential inclusion is identical to the PC flow defined by f, but it may have more stationary vertices. However, the extra stationary vertices are of zero index since they have two hyperbolic and no elliptic sectors [Ear99]. We conclude that the fixed point index of f on a Morse set is the same as the fixed point index of g on the corresponding Morse set.

Appendix D: Detailed runtime statistics

We collect the detailed information on each test run described in Section 7 in Tables 1 and 2.

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