1. Derivation of Algorithm 1

Each coefficient \( c_k \) in Eq. (11) in the main paper is given by a path integral of the form

\[
c_k^\Omega (\Omega, \sigma) = \int_{\Omega_k} \frac{f(x)}{p(x)} \, dx,
\]

where \( \Omega_k \) denotes the space of light transport paths with exactly \( k \) volumetric scatterings. Then, it can be shown \([KSZ\r^{15}]\) that

\[
\frac{\partial c_k^\Omega}{\partial \sigma} (\Omega, \sigma) = \int_{\Omega_k} \frac{f'(x)}{p(x)} \, dx,
\]

where \( f' \) is the partial derivative of \( f \) with respect to \( \sigma \).

When performing our symbolic path tracing (Algorithm 1) to estimate the coefficients \( c_k^\Omega \), the path throughput \( \gamma \) effectively keeps track of \( f/p \) when incrementally constructing the light transport path \( x \). Thus, Eq. (2) can be estimated by \( \gamma = \frac{f'(x)}{f(x)} \).

Notice that \( f(x) = \sigma^k \exp(-\sigma) f_0(x) \), where \( k \) is the number of volumetric scatterings, \( \ell \) denotes the total length of path segments inside the grain, and \( f_0 \) captures all the other terms independent of grain optical density \( \sigma \) (e.g., BSDF and phase functions). Then, it can be verified that

\[
\frac{\partial f}{\partial \sigma} = \frac{f'(x)}{f(x)} T = \frac{\frac{\partial}{\partial \sigma} \left[ \sigma^k \exp(-\sigma) f_0(x) \right]}{\sigma^k \exp(-\sigma) f_0(x)} = \frac{k}{\sigma} - \ell \cdot T \tag{3}
\]

giving Line 26 of Algorithm 1.

2. Extended Polynomials

Fitting \( \tau \). In §3.4, we face the problem of finding \( \tau > 0 \) such that \( c_k \approx c_k \exp(\tau(K-k)) \) for all \( K < k \leq K' \). This problem can be formulated as minimizing the \( L^2 \) difference in the logarithmic space:

\[
E(\tau) := \sum_{i=1}^{K'-K} \left[ \log(c_{K+i}) - \log(c_k \exp(-\tau)) \right]^2
\]

\[
= \left( \sum_{i=1}^{K'-K} \frac{K'-K}{\tau^2} \right) \tau^2 - 2 \left( \sum_{i=1}^{K'-K} i \Delta c_i \right) \tau + \left( \sum_{i=1}^{K'-K} \Delta c_i^2 \right).
\]

To find the minimizer of Eq. (4), we simply differentiate \( E \) and set the derivative to zero, resulting in Eq. (22).

Extended polynomials for derivatives. The coefficients \( \{ \partial c_k \} \) representing the derivatives of the GSDF and volume rendering parameters (with respect to grain density \( \sigma \)) can be handled in a similar way. Specifically, our experiments indicate that they can be well approximated with polynomials of degree five: for all \( k > K \),

\[
\partial c_k \approx \sum_{i=0}^{29} u_i (K-K)^i
\]

for some \( u_0, u_1, \ldots, u_{29} \in \mathbb{R} \) (see Figure ??-b). Then, \( \sum_{n=K+1}^{\infty} \partial c_k \alpha^d = \alpha^K \sum_{i=0}^{29} u_i \sum_{n=K+1}^{\infty} n^d \alpha^d \), which can be calculated analytically since series of the form \( \sum_{n=1}^{\infty} n^d \alpha^d \) converges (as \( 0 \leq a < 1 \)) and has close-form solutions.

3. Validations of extended polynomials

In main paper §3.1, we presented an extended polynomials formulation to efficiently handle highly scattering grains. Figure 1 shows a white furnace test using renderings generated with proxy path tracing (PPT) and volume path tracing (VPT) based on polynomial-valued GSDF and medium scattering parameters, respectively. When truncating the polynomials at degree \( K = 30 \), the results suffer from severe energy loss. In contrast, when storing polynomial

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coefficients up to $K = 29$ plus an additional term $\tau$ (which results in the same storage), much better accuracy can be achieved.

4. Figures of full resolution

In this section, we provide another version of Figure 9, 11 and 12, with all images shown in full resolution.

References

Figure 2: Figure 9 in the main paper

Figure 3: Figure 11 in the main paper
Figure 4: Figure 12 in the main paper