Tutorial: Inverse Computational Spectral Geometry

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Partial shapes
Subregions of a given shape
Question 1

- What is the relation between the global spectrum and the spectrum of the partialities?
Question 1b

• What is the relation between the spectra of different partialities?
Question 2

• Can we solve inverse problems starting from the spectrum of partialities?
The spectrum of Partial shapes

Operator

Spectrum
Different possible operators

- Laplacian of the patch
  “Computing Discrete Minimal Surfaces and Their Conjugates”,

- Hamiltonian
  “Hamiltonian operator for spectral shape analysis”,
  Y. Choukroun et al. 2018.

- LMH
  “Localized Manifold Harmonics for Spectral Shape Analysis”,
  S. Melzi et al. 2018.
Partial Functional Maps (PFM)

Partial Laplacian eigenfunctions

Weyl’s law: The Laplacian spectrum has a slope inversely proportional to the surface area.

Partial Functional Maps (PFM)

\[
\min_{C \in \mathbb{R}^{k \times k}, \nu : X \to [0, 1]} \| CA - B(\nu) \| + \rho_{\text{corr}}(C) + \rho_{\text{part}}(\nu)
\]

Preserve the area of \( Y \) + smoothness

\[
\rho_{\text{corr}}(C) = \| C \circ W \|_F^2
\]

Remark

• Spectral quantities can be used to analyze partialities of 3D objects
Can we retrieve the partial shape that generates a given spectrum?
Correspondence-Free Region Localization for Partial Shape Similarity via Hamiltonian Spectrum Alignment

“A. Rampini et al. 2019.”
Background: Laplace-Beltrami operator

\[ \Delta \phi_i(x) = \mu_i \phi_i(x) \]
Background: Laplace-Beltrami operator

\[ \Delta \phi_i(x) = \mu_i \phi_i(x) \]
Background: Hamiltonian operator

\[
(\Delta + v(x)) f(x) = \Delta f(x) + v(x) f(x)
\]

Hamiltonian \( H \)  
Potential function

Background: Hamiltonian operator

\[(\Delta + v(x)) f(x) = \Delta f(x) + v(x)f(x)\]

Hamiltonian $H$ \quad Potential function

\[H\psi_i(x) = \lambda_i \psi_i(x)\]

Background: Hamiltonian operator

\[ H \psi_i(x) = \lambda_i \psi_i(x) \]

Step potential

\[ v_T : \mathcal{X} \rightarrow \{0, \tau\} \]

Our approach: main idea

**Theorem:** There exists a step potential for which the Hamiltonian on the full shape and the LBO on the partial shape share the same spectrum:
Optimization problem

$\Delta + \text{diag}(v)$
Optimization problem

\[ \lambda(\Delta + \text{diag}(v)) \]
Optimization problem

\[ \lambda(\Delta + \text{diag}(v)) - \mu \]
Optimization problem

$$\min_{v \geq 0} \| \lambda (\Delta + \text{diag}(v)) - \mu \|_w^2$$
Optimization problem

$$\min_{\mathbf{v} \geq 0} \|\lambda(\Delta + \text{diag}(\mathbf{v})) - \mu\|_w^2$$

$$\|\lambda - \mu\|_w^2 = \sum_{i=1}^{k} \frac{1}{\mu_i^2} (\lambda_i - \mu_i)^2$$
Optimization problem

\[
\min_{\mathbf{v} \geq 0} \| \lambda (\Delta + \text{diag}(\mathbf{v})) - \mu \|^2_w \\
\| \lambda - \mu \|^2_w = \sum_{i=1}^{k} \frac{1}{\mu_i^2} (\lambda_i - \mu_i)^2
\]

Implementation details:

- Optimize over \( \mathbf{v} \in \mathbb{R}^n \) with saturation \( \sigma(\mathbf{v}) = \frac{r}{2}(\tanh(\mathbf{v}) + 1) \)
- Trust Region
- Initialization: multistart
Pros

1. Correspondence-free

2. Invariance to isometries

3. No descriptors in the optimization

Partial Functional Maps (PFM) is the main competitor
Qualitative comparisons

PFM

0.38
0.96
0.62
0.96

0.39
0.79
0.48
0.77
Limitations and future directions

• Different local minima
• Strong dependency on the discretization
• Computation of the spectrum is not efficient

• Improve robustness to noise
• Consider other data
• Learning pipeline
Can we perform operations in the space of partial spectra?
Spectral Unions of Partial Deformable 3D Shapes

Abstract

Spectral geometric methods have brought revolutionary changes to the field of geometry processing — however, when the data to be processed exhibits severe partiality, such methods fail to generalize. As a result, there exists a big performance gap between methods dealing with complete shapes, and methods that address missing geometry. In this paper, we propose a possible way to fill this gap. We introduce the first method to compute compositions of non-rigidly deforming shapes, without requiring to solve first for a dense correspondence between the given partial shapes. We do so by operating in a purely spectral domain, where we define a union operation between short sequences of eigenvalues. Working with eigenvalues allows to deal with

Motivation

- **Localization** of the union on a given template
- Full **geometry reconstruction** from partial views
- **Shape retrieval** from partial views
Do we need correspondences?
Do we need correspondences?

Typical pipeline:

1. Find partial correspondence
Do we need correspondences?

Typical pipeline:

1. Find partial correspondence

2. Extract non-rigid transformation
Do we need correspondences?

Typical pipeline:

1. Find partial correspondence
2. Extract non-rigid transformation
3. Merge partial views into a consistent discretization
Isometry invariant representation
The Spectrum is the right tool

- Invariant to isometries
- Invariant to different representations
- Does not require a correspondence
Learning the spectral union

Spectral decoder

Spectral union

Laplacian eigenvalues
Spectral Union

$\Lambda_1 \rightarrow E \rightarrow D \rightarrow T_A \rightarrow E \rightarrow D$

$\Lambda_2 \rightarrow E \rightarrow D \rightarrow T_A \rightarrow E \rightarrow D$

$\Lambda_{M_1 \cup M_2} \rightarrow \text{mse}$

$T_A = \text{Transformer A}$

$T_B = \text{Transformer B}$
"Instant recovery of shape from spectrum via latent space connections", R. Marin et al., 2020.
Limitations and future directions

- Different local minima
- Strong dependency on the boundary
- Missing guarantee that the predicted sequences are eigenvalues

- Improve robustness to noise
- Injecting a spectral term in the loss
- Other operations
What is next?
Other data

Point clouds  Range map  Volumetric  Implicit

“Fast Parallel Surface and Solid Voxelization on GPUs”, M. Schwarz et al., 2010
“Implicit Geometric Regularization for Learning Shapes”, A. Gropp et al., 2020
General approach

1. Define an operator (Laplacian)

2. Study its eigenvalues

3. Analyze the variables that define the data

4. Write these variables as a function of the spectrum

5. Define a procedure to solve this function
Graphs

Road maps | Social networks | Molecules | Functional networks
Graph spectrum

$\lambda_0 = 0$

$\lambda_1 = 0.15$

$\lambda_2 = 0.25$

$\lambda_3 = 0.59$

$\lambda_4 = 0.95$

$\lambda_{18} = 5.27$
Graph spectrum
Is it possible to recover a graph from its eigenvalues?
"Generation of isospectral graphs", L. Halbeisen et al., 1999
A special case

Generalized eigenproblem:

\[ L x = \lambda M x \]

\[ M = \text{diag}(m_1, \ldots, m_n) \]

“Reconstruction of weighted graphs by their spectrum“, L. Halbeisen et al., 2000
Isospectral and Subspectral Molecules

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Isospectral molecules are non-identical structures which possess the same spectrum of eigenvalues. Methods for recognizing isospectrality, procedures of Hellbrunner, Herndon and Živković for constructing new isospectral mates, and the specification of the relationship among the eigenvectors of the adjacency matrix of isospectral pairs are discussed here.
Are graphs harder than shapes?

• Isospectralization recovers $3 \times N$ numbers from $k$ eigenvalues

• It works well if we parametrize the shape with $O(k)$ parameters

• Graphs are defined by $N^2$ numbers of the adjacency matrix
Open problems
Compute eigenvalues of an operator via eigensolvers (Lanczos method)
“Alien” Learnable eigenvalues computation

We can try to learn them via neural networks.
Why is it important?

- Standard methods are slow for optimization-online augmentation:
  - Standard eigensolver: ~0.3 s (~6K vertices – first order)
  - Standard eigensolver: ~48 s (~30K vertices – third order)

- We can use NNs as differentiable blocks in other pipelines
A first solution

“Instant recovery of shape from spectrum via latent space connections”, R. Marin et al., 2020.
Application: fast isospectralization

- Isospectralization requires gradient with respect to eigenvalues
- With solver is hard/slow to compute
- Use eigenvalue approximator

"Isospectralization, or how to hear shape, style, and correspondence", L. Cosmo et al., 2019
Some considerations

- Instability of higher frequencies: could be solved via hierarchical architectures
- Invariance to isometry must be imposed: $\phi(Tx) = \phi(x)$
- Training loss not clearly defined
Several local minima

Existence of several local minima in the isospectralization problem

Symmetries and Isometries
Space of meaningful shapes

The spectral information can lead out of the space of “real” shapes

Target  With prior  Without prior
Thanks!

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Outline

- Partialities and geometry processing
- Correspondence-free region localization
- Spectral Unions
- Other domains
- Open problems and Limitations