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Swept volumes of many poses

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Abstract

We consider the swept volume $\mathscr{A}(X)$ of a rigid body X which assumes a general set $\mathscr{A}$ of positions. A special case of this is a one-parameter motion of X, where the set of poses is curve-like. Here we consider a full-dimensional subset $\mathscr{A}$ of the motion group. Such a set of poses can be seen as the tolerance zone of an imprecisely defined pose. Alternatively, a set of poses may be obtained by by measurements or simulation. We analyze the geometric properties of such sets of poses and give algorithms for computing the boundary $\partial\mathscr{A}(X)$ in the case that $\mathscr{A}$ is a discrete pose cloud. The dimension of the problem, which equals six a priori, is reduced to two.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling

1. Introduction.

The volume swept by a moving rigid body is a topic of great interest and is extensively studied in the literature. We do not attempt to give an exhaustive list of references, but mention only [AMBJ] for an overview, [KVLM04] for computation, and [BL92] for some mathematical methods. The available literature deals mostly with one-parameter sweeps.

Speaking from a more general an abstract viewpoint, we could say that a rigid body X *moves* when it assumes any of a given set $\mathscr{A}$ of positions. We use *pose* as a synonym for position. The *swept volume* means the union of all positions $\alpha^i(X)$ of the rigid body X, as α^i runs through $\mathscr{A}$. We write $\mathscr{A}(X)$ for this swept volume.

An important special case of this concept is that X moves only by translations: The new position $\alpha(X)$ of the rigid body under consideration is the set $X+y$, where y is a translation vector taken from a set Y:

$$\mathscr{A}(X) = \{X+y \mid y \in Y\} = \{x+y \mid x \in X, y \in Y\} = X+Y.$$

We see that the swept volume coincides with the *Minkowski sum* $X+Y$ of the sets X and Y. Minkowski sums are an active area of research. The list of references given here [AFH02, EHS02, FH00, HH03, SH02, VM04] is by no means exhaustive.

If $\mathscr{A}$ is a one-parameter set, either in the discrete or the continuous sense, then X undergoes a one-parameter motion, moving from one pose to the next. An example of a higher-dimensional motion is provided by the Minkowski sum case above, where X moves by translations: If Y has interior points, then it has dimension three, and X undergoes a three-dimensional motion, assuming a three-dimensional set of positions in space. The present paper deals with a full-dimensional subset $\mathscr{A}$ of the Euclidean motion group, whose dimension equals six.

Such a set of poses can have the following two interpretations: One is that a pose α is imprecisely defined, and the amount of uncertainty is specified by a *tolerance zone* $\mathscr{A}$, which is a neighbourhood of α. The other interpretation is that X undergoes a small unstructured motion, and poses $\alpha^1, \alpha^2, \ldots$ have been obtained by measurements or simulation. This collection of poses then is a point cloud-like object (a *pose cloud*), whose shape is that of a 6-dimensional subset of the Euclidean motion group.

The continous case and the disrete case.

There is a continuous version of the concepts mentioned above (rigid body, set of poses, swept volume), and also a discrete one. For computational purposes, the rigid body X is represented by its boundary as triangle mesh, and the set

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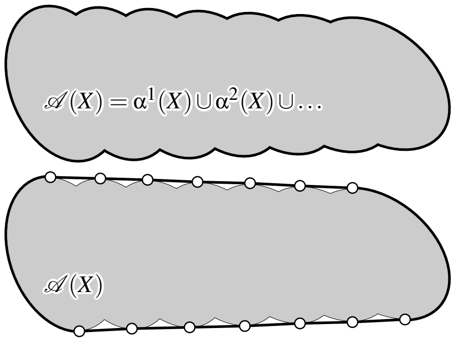


Figure 1: *The difference between Boolean union (top) and envelope (bottom) in the case of a discrete 1-parameter motion. Differences are in the smoothness of the swept volume's boundary $\partial\mathscr{A}(X)$ and the computational cost.*

$\mathscr{A}$ of poses by a *pose cloud*. The swept volume will be given by a triangle mesh again. Computationally, there are two approaches to computing the swept volume, which for the 1-dimensional case are illustrated by Fig. 1.

If the given poses are $\alpha^1, \alpha^2, \dots$, we can compute the Boolean union $\alpha^1(X) \cup \alpha^2(X) \cup \dots$. The result is an approximation (e.g., via a triangle mesh) of the volume $\mathscr{A}(X)$. As Fig. 1 clearly shows, the smoothness of the volume computed in this way often does not adequately reproduce the smoothness of the volume $\mathscr{A}(X)$. Boolean union not only results in insufficient smoothness, also its computational cost is high. It is therefore often important to be able to find candidates for the boundary points of $\mathscr{A}(X)$ without having to resort to Boolean set operations. Thus one is led to consider the *envelope* of a moving surface (the boundary of X) with respect to a smooth motion. This approach works well if both the body X and the set of poses $\mathscr{A}$ are at least piecewise smooth. The present paper approaches the computation of $\mathscr{A}(X)$ via envelopes.

The relation to tolerance analysis

The concept of tolerance zone which represents an imprecisely defined object [Req83, HV95, Hof01] has been used in a geometric context e.g. in [POP*00] and [WKP00], where geometric constructions occurring in Computer-Aided geometric design are analyzed from the tolerancing viewpoint. Tolerance zones for motions are studied in [SW05] from an abstract point of view. There is also related work on geometric transformations in the 2D case [FMR00, FMR01, FP02].

Within the tolerance analysis context, the present paper solves the worst case tolerancing problem of computing a bounding volume for the position $\alpha(X)$ of a rigid body X, where the pose α is only known to be contained in some set $\mathscr{A}$. X itself may already be the tolerance zone of a point.

Applications: Computing bounding volumes

The sequential nature of time does not allow genuine multi-parameter motions to take place in the real world. However, there are situations where a rigid body executes a one-parameter motion of a complicated, chaotic, or unknown nature, and nevertheless one is interested in a bounding volume which contains all possible positions $\alpha(X)$.

In that case measurements or simulation may provide a collection of poses which more or less densely covers a certain subset $\mathscr{A}$ of the Euclidean motion group. The latter has dimension six, so the dimension of $\mathscr{A}$ can be any of $0, \dots, 6$. In this paper we are not concerned with the issue of estimating that dimension. We consider the full-dimensional case and are aware of the fact that pose clouds can be "thin" and thus represent lower-dimensional shapes.

Overview.

We first present elementary Euclidean kinematics in Sec. 2: poses, velocities, and infinitesimal motions. Because we later need them for theoretical investigations, also the matrix exponential function and logarithm are introduced. Sec. 3 deals with tolerance zones $\mathscr{A}$ of poses, i.e., full-dimensional subsets of the Euclidean motion group, and with the question what happens if a rigid body X assumes all poses in $\mathscr{A}$. We consider the abstract question of outward normal vectors of tolerance zones and derive a theoretical result on the oriented envelope of a rigid body X with respect to $\mathscr{A}$. Sec. 4 deals with pose clouds, their support planes, and the actual computation of the swept volume, in part using the matrix logarithm. In Sec. 4.3 we show how to avoid the matrix logarithm in computations. We further consider a smoothing process which takes the tolerancing side conditions into account. Numerical examples (Sec. 6) conclude the paper.

2. The Euclidean motion group.

In Sec. 2 we present facts about kinematics and its relations to line geometry which can be found e.g. in [BR79] or [PW01].

The position of a rigid body X in 3-dimensional Euclidean space is given by an orthogonal matrix A and a translation vector a. We write $\alpha = (A, a) \in \mathbb{R}^{3\times3+3}$ to indicate such a position. If X assumes position α, it is moved to $\alpha(X)$, which means that $x \in X$ is transformed to the point $y = Ax + a$. We do not consider orientation-reversing poses, so we forbid $\det A = -1$ and require $\det A = 1$. The Euclidean motion group SE_3 is the set of such poses. It is a six-dimensional surface in the space $\mathbb{R}^{3\times3+3}$ of matrix/vector pairs.

We further use the following property of skew-symmetric matrices: For any skew-symmetric 3 by 3 matrix V, there is a vector c such that $Vx = c \times x$ for all x. The corresponding notation is as follows:

$$c = (c_1, c_2, c_3),\ V = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix} \iff \begin{cases} c = axis(V), \\ V = Skew(c). \end{cases}$$

2.1. Smooth motions and their velocities.

With the real parameter t as time, a smooth motion $\alpha(t) = (A(t), a(t))$ consists of a matrix-valued smooth function $A(t)$ and a vector-valued smooth function $a(t)$ such that $\alpha(t)$ is a pose in SE_3 for all t. The trajectory of the point x under this smooth motion is the curve $\alpha(t) \cdot x = A(t)x + a(t)$. The smooth motion itself can be seen as a curve lying in SE_3.

The velocity vector of the point x is the derivative

$$\dot\alpha(t) \cdot x = \frac{d}{dt}(A(t)x + a(t)) = \dot A(t)x + \dot a(t), \tag{1}$$

but we also employ the *velocity with respect to the coordinate system attached to X*. This means the vector $v_t(x)$ such that $A_t(t)v_t(x)$ equals the velocity vector of x:

$$v_t(x) = A(t)^{-1}\dot A(t)x + A(t)^{-1}\dot a(t). \tag{2}$$

Because the matrix $A(t)^{-1}\dot A(t)$ is skew-symmetric, we can define two vectors $d, \bar d$ by

$$v_t(x) = d(t) \times x + \bar d(t), \quad \text{where} \tag{3}$$

$$\bar d(t) = A(t)^{-1}\dot a(t),\ d(t) = axis(A(t)^{-1}\dot A(t)). \tag{4}$$

It is convenient to identify poses and their derivatives with block matrices as follows:

$$\alpha(t) \equiv \begin{bmatrix} 1 & 0 \\ a(t) & A(t) \end{bmatrix}, \quad \dot\alpha(t) \equiv \begin{bmatrix} 0 & 0 \\ \dot a(t) & \dot A(t) \end{bmatrix}. \tag{5}$$

Now that poses are matrices, we can multiply and invert them. It is elementary that $(A,a)\cdot(B,b) = (AB, Ab + a)$ and $(A,a)^{-1} = (A^{-1}, -A^{-1}a)$, with $A^{-1} = A^T$. Further, the vectors $d(t), \bar d(t)$ of (3) fulfill the relation

$$\begin{bmatrix} 0 & 0 \\ \bar d(t) & Skew(d(t)) \end{bmatrix} = \alpha(t)^{-1}\dot\alpha(t). \tag{6}$$

Observe that the vectors $d, \bar d$ do not change if α and $\dot\alpha$ are replaced by $\beta\alpha$ and $\beta\dot\alpha$ for any pose β.

2.2. Velocities and the tangent spaces of SE_3.

Any surface M has a tangent space in each of its points. It consists of the derivative vectors of curves in the surface which pass through that point. For the surface SE_3, points are poses, and curves are smooth motions. A time-dependent pose $\alpha(t)$, either seen as a matrix/vector pair, or as a block matrix in the sense of (5), has a derivative $\dot\alpha(t)$, which either is seen as a matrix/vector pair, or as a block matrix according to (5). The derivative $\dot\alpha = (\dot A, \dot a)$ is called an *infinitesimal motion* attached to the pose $\alpha = (A, a)$. For each pose α, there is a six-dimensional space of infinitesimal motions attached to α. We use the vectors $d, \bar d$ computed with (6) or (4) as coordinates for infinitesimal motions. Thus the six-dimensional abstract tangent space of SE_3 at a given pose is identified with the space of $d, \bar d$'s.

Recall that a straight line parallel to the vector l which passes through the point x is assigned the Plücker coordinates $l, \bar l$ with $\bar l = x \times l$. These coordinates have the property that $\bar l$ does not depend on the choice of x on the line, and the line is recovered from the coordinates $l, \bar l$ as the solution set of the three linear equations $\bar l = x \times l$ in the variable x. Any pair $l, \bar l$ with $\langle l, \bar l\rangle = 0$ and $l \neq 0$ occurs as Plücker coordinates of a line in Euclidean three-space.

If a body X in three-space has a smooth boundary, we can select a boundary point x and consider an outward normal vector n there. The line orthogonal to the boundary in the point x (the surface normal) has the Plücker coordinates $n, \bar n$ with $\bar n = x \times n$ according to the previous paragraph. Choose a pose $\alpha = (A, a)$. Then the outward normal vector of $\alpha(X)$ at the boundary point $Ax + a$ is given by An. We are interested in infinitesimal motions attached to the pose α which move x towards the inside of $\alpha(X)$.

The infinitesimal motion $\dot\alpha$ does not move x towards the outside $\alpha(X)$, if and only if the velocity vector $\dot\alpha \cdot x$ of (1) does not point towards the outside of $\alpha(X)$. With the normal vector n, this relation is expressed by

$$\langle \dot A x + \dot a, An\rangle \leq 0. \tag{7}$$

When using coordinate vectors $d, \bar d$ for the infinitesimal motion, and the Plücker coordinates $n, \bar n$ for the surface normal, this is equivalent to

$$\langle d, \bar n\rangle + \langle \bar d, n\rangle \leq 0. \tag{8}$$

(as follows from $\langle \dot A x + \dot a, An\rangle = \langle A^{-1}\dot A x + A^{-1}a, n\rangle = \langle d \times x + \bar d, n\rangle$.)

Remark: The velocity vector of x is *tangent* to the boundary of $\alpha(X)$ if and only if $\langle d, \bar n\rangle + \langle \bar d, n\rangle = 0$ holds. This is the condition familiar from kinematics that the line with Plücker coordinates $n, \bar n$ is a path normal of the infinitesimal motion $\dot\alpha$.

2.3. Straightening SE_3.

A parametrization of the surface SE_3 is given by the matrix exponential function: A pose depends on $(v, \bar v) \in \mathbb{R}^{3+3}$ via

$$\alpha(v, \bar v) = \exp\begin{bmatrix} 0 & 0 \\ \bar v & Skew(v) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a(v, \bar v) & A(v, \bar v) \end{bmatrix}. \tag{9}$$

We use the notation $\alpha = \exp(v, \bar v)$, $(v, \bar v) = \log\alpha$. For the actual computation "exp" and "log" see the appendix. It is well known that "exp" maps the domain defined by $\|v\| < \pi$ diffeomorphically onto the set of poses whose rotation angle is less than π.

For measuring the distortion when taking the logarithm we use the Frobenius norm of a matrix defined by $\|M\|^2 := \mathrm{tr}(M^T M)$. If $\alpha = (A, a)$ is a pose and $\dot\beta$ with coordinates

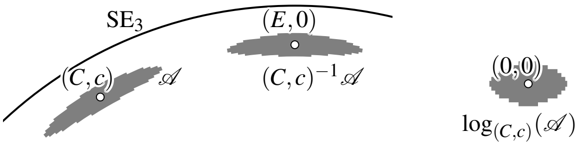


Figure 2: *Schematic illustration of a tolerance zone $\mathscr{A}$ in* SE_3*, poses* (C,c) *and* $(E,0)$*, and the matrix logarithm.*

$d,\bar{d}$ is an infinitesimal motion, the block matrices which represent α and β have the norms $\|\dot{\beta}\|^2 = 2\|d\|^2 + \|\bar{d}\|^2$, $\|\alpha\|^2 = 4 + \|a\|^2$. Near the identity pose $(E,0)$, we have the approximate identity

$$\exp(v,\bar{v}) \approx (E + Skew(v), \bar{v}), \tag{10}$$

which is made more precise below. This means that near the identity we may use $v,\bar{v}$ as coordinates for poses, and we may use the matrix logarithm (at least theoretically) for flattening SE_3 and analyzing small subsets of it.

The above approximation is more precisely expressed by the inequality

$$\|\exp(v,\bar{v}) - (E + Skew(v), \bar{v})\| \le g(R), \tag{11}$$
$$\text{where } g(t) = e^t - 1 - t,\ R^2 = 2\|v\|^2 + \|\bar{v}\|^2,$$

as shown in the appendix. The function $g(t)$ has $g(0) = \dot{g}(0) = 0$, so the approximation is very good if both $v,\bar{v}$ are small. For bigger $v,\bar{v}$, this inequality gives only little information, because $g(t)$ grows rapidly.

The following well known property of the logarithm is an easy consequence of the previous inequality:

Proposition 1 *If $\alpha(t)$ is a smooth one-parameter motion which passes through the identity pose $(E,0)$ for $t = 0$ and has the tangent vector (i.e., infinitesimal motion) with coordinates $d,\bar{d}$ there, then also the curve $\log\alpha(t)$ in $\mathbb{R}^6$ has the tangent vector $(d,\bar{d}) \in \mathbb{R}^6$ at $t = 0$.*

For straightening a piece of SE_3 around a pose α, we use

$$\log_\alpha(\beta) := \log(\alpha^{-1}\beta). \tag{12}$$

(12) is a way to represent poses near α by vectors in $\mathbb{R}^{3+3}$. A domain where $\log_\alpha$ can be unambiguously defined is e.g. the set of poses β where the rotation angle between α and β is less than π. The mapping $\log_\alpha$ is schematically illustrated by Fig. 2.

3. Tolerance zones.

It is an aim of this paper to deal with discrete "pose clouds". Like in the case of $\mathbb{R}^3$, where point clouds represent solids or surfaces, pose clouds represent six-dimensional solids in SE_3. We first have a look at the continuous case, i.e., the case of a domain with smooth boundary inside SE_3. Later we consider pose clouds which represent such solids.

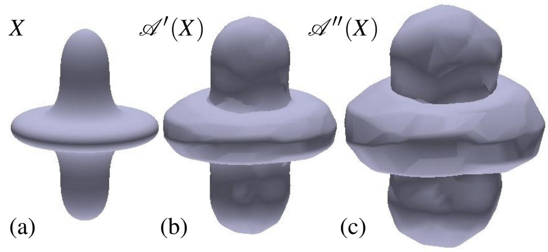


Figure 3: *From left to right: The sets X, $\mathscr{A}'(X)$, and $\mathscr{A}''(X)$ for different non-smooth tolerance zones. The diameters of $\mathscr{A}'$ and $\mathscr{A}''$ are 0.4 and 0.8, respectively.*

Suppose that $\mathscr{A}$ is such a set of poses in SE_3. We assume that $\mathscr{A}$ is the closure of its interior (topological properties refer to the manifold SE_3, not to ambient space $\mathbb{R}^{3\times3+3}$) and is compact.

3.1. Swept volumes.

The swept volume $\mathscr{A}(X)$ of a rigid body X which assumes every pose in the set $\mathscr{A}$ is defined as the union of all $\alpha(X)$ as α ranges in $\mathscr{A}$. Such volumes are illustrated in Fig. 3. We are interested in the boundary $\partial\mathscr{A}(X)$. The following elementary statement, which is a first step in this direction, uses the boundaries ∂X and $\partial\mathscr{A}$ of the bodies X and $\mathscr{A}$.

Proposition 2 *For a point $x \in X$ and a pose $\alpha \in \mathscr{A}$, the point $\alpha \cdot x$ is contained in the boundary $\partial\mathscr{A}(X)$ of the swept volume only if x is a boundary point of X and the pose α is a boundary pose of $\mathscr{A}$.*

As has been remarked in the introduction, the computing of Minkowski sums could be seen as a special case of this paper, if all motions are translations. Prop. 2 has a counterpart in the Minkowski sum context: If $x \in X$ and $y \in Y$, then $x + y$ is a boundary point of $X + Y$ only if both $x \in \partial X$ and $y \in \partial Y$.

Proof: (i) If x is not a boundary point of X, then neither is $\alpha \cdot x$ a boundary point of $\alpha(X)$, regardless of α. It follows that $\alpha \cdot x$ is no boundary point of $\mathscr{A}(X)$. (ii) If α is in $\mathscr{A}$, but not in the boundary $\partial\mathscr{A}$, then small translations in all directions will change α such that it is still contained in $\mathscr{A}$. It follows that for any x, $\alpha \cdot x$ can still be translated in all directions without leaving $\alpha(X)$. Thus it is no boundary point of the swept volume. □

3.2. Tangent spaces of tolerance zones

The boundary surface $\partial\mathscr{A}$ of the tolerance zone $\mathscr{A}$ has five-dimensional tangent spaces. The tangent space at the pose α is a subspace of the six-dimensional space of infinitesimal motions attached to α. Fortunately our introduction of coordinates $d,\bar{d}$ for infinitesimal motions by (3) identifies the space of infinitesimal motions attached to a pose α with the vector space $\mathbb{R}^{3+3}$ of pairs $d, \bar{d}$, so a five-dimensional subspace is determined by *one* linear relation between the six coordinates of $d, \bar{d}$: We are numbering the coordinates of $d, \bar{d}$ such that $d = (d_1, d_2, d_3)$ and $\bar{d} = (d_4, d_5, d_6)$. The *coefficients* in the linear relation are numbered in an unorthodox way:

$$n_4 d_1 + n_5 d_2 + n_6 d_3 + n_1 d_4 + n_2 d_5 + n_3 d_6 = 0. \qquad (13)$$

We collect the coefficients n_i in two vectors $n, \bar{n}$ such that $n = (n_1, n_2, n_3)$ and $\bar{n} = (n_4, n_5, n_6)$. Then (13) reads

$$\langle \bar{n}, d \rangle + \langle n, \bar{d} \rangle = 0. \qquad (14)$$

3.3. Flattening tolerance zones in SE_3.

The reason why we apply mappings like the logarithm to poses is that a vector space is a friendly environment with regard to computing tangent spaces and their linear equations. Moreover, the logarithm has the following nice property:

Proposition 3 *The equation of the boundary's tangent space is the same for α in $\mathscr{A}$ and for 0 in $\log_\alpha(\mathscr{A})$.*

Proof: The coordinates $d, \bar{d}$ for an infinitesimal motion $\dot{\beta}$ attached to α do not change if we multiply both $\dot{\beta}$ and α with the same pose from the left. Thus $\mathscr{A}$ has at α the same tangent space equation as $\mathscr{B} := \alpha^{-1} \cdot \mathscr{A}$ has at $\alpha^{-1} \cdot \alpha$.

By Prop. 1, taking the logarithm does not change the coordinates of tangent vectors. So if the identity pose happens to be a boundary pose of $\mathscr{B}$, then $\log_\alpha(\mathscr{A}) = \log(\mathscr{B})$ has in $(0,0)$ the same tangent vectors as $\mathscr{B}$ has in $(E, 0)$. $\square$

3.4. Envelopes.

Sec. 3.4 contains the main theoretical results of this paper. We extend the concept of *normal vector pointing outwards* which is well known in the context of smoothly bounded solids to tolerance zones. We define the *oriented envelope* of a rigid body with respect to a tolerance zone and show that the boundary of the swept volume is contained in this envelope. By the passage to so-called *outer part*, equality is achieved. This is the basis of our algorithms given later — we compute the boundary of the swept volume via computing the oriented envelope.

The well known Minkowski sum case

if X and Y are bodies in $\mathbb{R}^3$ with a smooth boundary, then boundary points $x \in \partial X$ and $y \in \partial Y$ can contribute to a boundary point $x + y$ of the Minkowski sum $X + Y$ only if the tangent spaces of X at x and of Y at y are parallel. This is the so-called *envelope condition*. If it is possible to query Y for boundary points whose tangent plane has a given orientation, computation of the Minkowski sum's boundary is two-dimensional in nature: For a sample of boundary points $x^1, x^2, \ldots$ of X, we search for corresponding points in Y and thus get a surface-shaped collection of points. It is called the *envelope* of the boundary ∂X with respect to the translations defined by the boundary ∂Y. The actual boundary of $X + Y$ is contained in that surface. Another name for the envelope is *convolution surface* of the boundaries ∂X and ∂Y.

Without much effort it is possible to refine the envelope condition: Each boundary point of either of X and Y is given a normal vector which points towards the outside. Then $x + y$ is a boundary point of $X + Y$ only if the outward normal vectors associated with the points x and y coincide. Again, for a sample $x^1, x^2, \ldots$ of boundary points in X we can query the boundary of Y for points $y^{i,j}$ such that x^i and $y^{i,j}$ has the same normal vector. The boundary of the Minkowski sum is contained in the *oriented envelope* of X with respect to Y, which is the surface which contains all sums $x^i + y^{i,j}$. The envelope usually is twice as big as the oriented envelope.

It is the purpose of the following sections to generalize these concepts to sets of poses.

Outward normal vectors.

In general, the vector n is an *outward normal vector* of a solid in a boundary point, if for all vectors v which do *not* point toward the outside of X in that point, we have $\langle n, x \rangle \leq 0$. For a tolerance zone $\mathscr{A}$ (which is not a solid in a vector space) we do the following: In view of Prop. 3, the tangent space of $\mathscr{A}$ at a boundary pose α occurs also as tangent space of $\log_\alpha \mathscr{A}$. When grouping the coefficients in the linear equation of this tangent space as in (14), $(\bar{n}, n)$ is a normal vector of $\log_\alpha \mathscr{A}$. By multiplying both n and $\bar{n}$ with -1 if necessary, we can make the vector $(\bar{n}, n)$ point outward, and we say it is an *outward normal vector* of $\mathscr{A}$. The fact that $(\bar{n}, n)$ points outward means that for all vectors $(d, \bar{d})$ pointing inwards, we have

$$d_1 n_3 + d_2 n_4 + d_3 n_5 + d_4 n_1 + d_5 n_2 + d_6 n_3 \leq 0. \qquad (15)$$

As the boundary of the swept volume is two-dimensional, and the boundary of a tolerance zone has dimension five, only a small part (in fact, a two-dimensional one) can be expected to contribute to the boundary of the swept volume. With the solid X, this is different: Its boundary already has the right dimension, so we can expect that a substantial part of ∂X contributes to $\partial \mathscr{A}(X)$. Below follows a nice geometric relation between normal vectors of $\mathscr{A}$ and those poses which contribute to the swept volume's boundary.

Oriented Envelopes.

Def. 1 defines the concept of oriented envelope of a solid with respect to a full-dimensional set $\mathscr{A}$ of poses (its computation is the topic of Sec. 4). The purpose of this definition is to find a set which is not much larger than the boundary of the swept volume we are looking for.

Definition 1 Suppose that x is a boundary point of X with outward normal vector n. If $(\bar{n}, n)$ with $\bar{n} = x \times n$ is an outward normal vector of the tolerance zone $\mathscr{A}$ at the boundary pose β, then $\beta \cdot x$ is a point of the oriented envelope of X with respect to $\mathscr{A}$.

Proposition 4 *The boundary of the swept volume $\mathscr{A}(X)$ is contained in the oriented envelope of X with respect to $\mathscr{A}$.*

Proof: We assume that x, n and β are as in Def. 1. The solid $\beta(X)$ is contained in the swept volume $\mathscr{A}(X)$ and touches $\partial\mathscr{A}(X)$ from the inside in the point $\beta \cdot x$. Any smooth one-parameter motion $\alpha(t)$ which starts with $\alpha(0) = \beta$ and has $\alpha(t) \in \mathscr{A}$ for all t moves X inside the swept volume. So the velocity vector $\dot{\alpha} \cdot x$ at $t = 0$ points towards the inside of $\mathscr{A}(X)$, and therefore towards the inside of $\beta(X)$. If we use coordinate vectors $d, \bar{d}$ for the infinitesimal motion $\dot{\alpha}$, this fact is expressed by the inequality (8). This is the same inequality as (15) which says that $(\bar{n}, n)$ is an outward normal vector. □

The outer boundary of a solid.

In the context of this paper we are not interested in any interior holes the compact solids X and $\mathscr{A}(X)$ may have. We therefore employ the concept of *outer boundary*: For any compact set Y, the difference set $\mathbb{R}^n \setminus Y$ has exactly one unbounded component (the *outside* of Y). The part of the boundary of Y which is adjacent to the outside of Y is called the *outer boundary* of Y. If Y is a *surface*, then $\partial^{\mathrm{out}} Y$ exists, but we call it *outer part* of Y in order not to apply the word "boundary" to something which is boundary-shaped already.

The operation of computing the outer part of a surface is e.g. built in software which handles triangle meshes. It consists of the trimming away of interior surface components.

Proposition 5 *If X is a solid and $\mathscr{A}$ is a tolerance zone, then the outer boundary of the swept volume is the same as the outer part of the oriented envelope.*

Proof: The implication $\partial X \subset Y \subset X \implies \partial^{\mathrm{out}} X = \partial^{\mathrm{out}} Y$ is obvious from the definition of ∂^{out}. With Y as the oriented envelope, the result follows from Prop. 4. □

If we specialize this result to the case of Minkowski sums, we get the statement that $\partial^{\mathrm{out}}(X + Y)$ is the same as the outer part of of the convolution surface of ∂X and ∂Y.

All normal vectors occur.

If M is a compact smooth surface in Euclidean space, it is easy to show that every unit vector n occurs as an outward normal vector in some point x (choose the point x in M where $\langle x, n \rangle$ is maximal). With tolerance zones in SE_3, such simple arguments are not available, as the meaning of 'normal vector' is different and depends on the coordinates we have introduced for infinitesimal motions. There is however the following property of tolerance zones of simple shape, whose proof (given in the appendix) uses a topological argument.

Proposition 6 *Assume that the tolerance zone $\mathscr{A}$ is smooth, has the topology of a ball, and is contained in a subset of SE_3 where the mapping $\log_\alpha$ is well defined, for some α. Then for every unit vector $(\bar{n}, n) \in \mathbb{R}^{3+3}$ there is $\beta \in \partial\mathscr{A}$ such that $(\bar{n}, n)$ is an outward normal vector at the pose β.*

4. Point clouds and envelope computation.

We now consider pose clouds in SE_3, which are still denoted by $\mathscr{A}$. The poses contained in $\mathscr{A}$ are denoted by the symbols α^1, α^2 and so on. Alg. 2 given below employs the matrix logarithm, which means higher computational complexity than necessary. Sec. 4.3 shows how to get rid of logarithms.

4.1. Normal vectors of point clouds.

The vector n is called an outward normal vector of a *convex* point cloud $x^1, \dots, x^r$ in a vertex x^{i_0}, if $\langle x^{i_0}, n \rangle \geq \langle x^i, n \rangle$ for all i. This means that the entire cloud is contained in the halfspace with equation $\langle n, x \rangle \leq \langle n, x^{i_0} \rangle$. This halfspace is bounded by a *support plane* of the cloud. Of course, if the point cloud is dense and approximates a smooth surface, a normal vector defined in this way approximates the normal vector in the sense of differential geometry. For a given point cloud and normal vector n, there is always a vertex where this vector is an outward normal vector.

For a non-convex point cloud $\mathscr{A}$ this is no longer true. However, if we choose n from a uniform sample of points in the unit sphere and compute corresponding half-spaces which contain $\mathscr{A}$, then the intersection of those half-spaces approximates $\mathscr{A}$'s convex hull. The domain associated with the cloud in this way is not smaller than the domain represented by the cloud itself, and it is close to it if $\mathscr{A}$ happens to have convex shape. We collect the instructions for computing this *approximate convex hull* together with the points where given vectors are outward normal vectors in the following algorithm:

Algorithm 1 Suppose $x^1, \dots, x^r$ is a point cloud, and $n^1, \dots, n^s$ is a point cloud representing the surface of the unit sphere. Compute all values $\langle x^j, n^i \rangle$ and for each i, choose an index $j(i)$ such that $\langle x^{j(i)}, n^i \rangle \geq \langle x^j, n^i \rangle$ for all j. Then the vertex $x^{j(i)}$ has n^i as an outward normal vector, and the intersection of the half-spaces $\langle x, n^i \rangle \leq \langle x^{j(i)}, n^i \rangle$ is an approximate convex hull of the point cloud. ◇

4.2. Normal vectors of pose clouds.

We cannot apply Alg. 1 to a pose cloud $\mathscr{A}$ directly. But by definition, $(\bar{n}, n)$ is an outward normal vector at the boundary pose α, if it is an outward normal vector of $\log_\alpha \mathscr{A}$ at the origin of the coordinate system. This property can be used for *testing* if given $\bar{n}, n$ and α fulfill the normal vector condition. Searching for α when only $\bar{n}, n$ are given, is done in a way similar to Alg. 1, using the fact that the matrix logarithm has low distortion for small pose clouds.

Suppose that a rigid body is triangulated, with vertices x^i and outward normal vectors n^i at x^i. We compute Plücker

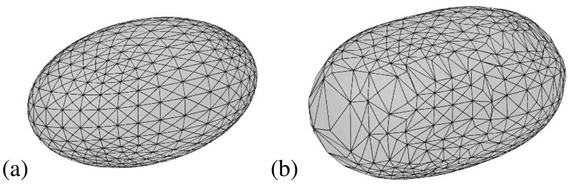


Figure 4: *(a): Triangle mesh representing the boundary of an ellipsoid X. (b): The boundary of a swept volume $\mathscr{A}(X)$.*

coordinates $n^i, \bar{n}^i$ with $\bar{n}^i = x^i \times n^i$. For each i, we want to find a pose $\alpha^{j(i)}$ of the given pose cloud where $(\bar{n}^i, n^i)$ is an outward normal vector. Similar to Alg. 1, we do not search the entire pose cloud, but a convex hull-like object associated with the pose cloud. If $\mathscr{A}$ represents a tolerance zone, this operation means convexification and thus enlarging the tolerance zone, i.e., an error on the safe side. We propose the algorithm below, which does the following: We take any logarithm of $\mathscr{A}$ and look for a pose where the given vector is a normal vector. This is only an approximate answer, however. So we now take the logarithm with respect to the pose thus found, and repeat the process until it becomes stationary.

Algorithm 2 Suppose a pose cloud $\alpha^1, \dots, \alpha^r$ and vectors $\bar{n}, n$ are given. Compute poses where $(\bar{n}, n)$ is an outward normal vector of the pose cloud as follows:

1. Let $N = 0$ and choose an index i_0 with $1 \le i_0 \le r$.
2. Compute the point cloud $\mathscr{V} = \log_{\alpha^{i_N}}(\mathscr{A})$, which consists of $((v^1, \bar{v}^1), \dots (v^r, \bar{v}^r))$. By construction, $(v^{i_N}, \bar{v}^{i_N}) = 0$.
3. Find $i_{\max}$ such that $\langle n, \bar{v}^i \rangle + \langle \bar{n}, v^i \rangle$ is maximal for $i = i_{\max}$.
4. Let $i_{N+1} = i_{\max}$, increment N.
5. If the sequence of indices computed has become periodic with period k (i.e., $i_N = i_{N-k}$), terminate with the output $i_{N-k}, \dots, i_{N-1}$. Otherwise continue with 2. $\diamond$

If Alg. 2 terminates with a unique index i_N, we have found a pose α^{i_N} where $(\bar{n}, n)$ is an outward normal vector. If the algorithm terminates with a periodic sequence $i_N = i_{N+k}$ with $k > 1$, there are k candidates for that pose. Which to choose, is the topic of Sec. 4.4 below. Before that, we given an elementary interpretation of Alg. 2.

4.3. An elementary interpretation.

In the proof of Prop. 4 we encountered the following situation: A pose β in $\mathscr{A}$ and a boundary point x of X with outward normal vector n have the property that $\beta \cdot x$ is a boundary point of the swept volume $\mathscr{A}(X)$. Then necessarily $\beta(X)$ touches the boundary of $\mathscr{A}(X)$ from the inside. Any velocity vector $\dot{\alpha} \cdot x$ attached to β which points towards the inside of $\mathscr{A}$ must fulfill

$$\langle \dot{\alpha} \cdot x, Bn \rangle \le 0 \quad (\beta = (B, b)). \tag{16}$$

As explained in that proof, this expresses the fact that any one-parameter motion inside $\mathscr{A}$ which starts in β assigns a velocity vector to x which points towards the inside of the swept volume. The inequality (16) also expresses the fact that $(n \times x, n)$ is an outward normal vector of $\mathscr{A}$.

Now $\mathscr{A} = \alpha^1, \dots, \alpha^r$ is a pose cloud. Assume that $\beta = \alpha^{i_0}$. All difference vectors $\alpha^i - \beta$ are vectors attached to β pointing towards the inside of $\mathscr{A}$. The denser $\mathscr{A}$, the better the set of difference vectors approximates the set of vectors pointing towards the inside.

It is easy to set up an algorithm which for given x finds a boundary pose β such that (16) is fulfilled. In view of the discussion above, this is in principle the same as Alg. 2, which finds a pose such that $(x \times n, n)$ is a normal vector of $\mathscr{A}$ at that pose. It goes as follows: First, (16) is rewritten as

$$\langle Bn, (A^i - B)x + (a^i - b) \rangle \le 0 \quad (i = 1, \dots, r). \tag{17}$$

This is equivalent to $\langle n, B^{-1}(A^i - B)x + B^{-1}(a^i - b) \rangle \le 0$, and in view of $\beta^{-1}\alpha^i = (B^{-1}A^i, B^{-1}(a^i - b))$ also equivalent to

$$\langle n, \beta^{-1}\alpha^i \cdot x \rangle \le \langle n, x \rangle \quad (i = 1, \dots, r). \tag{18}$$

Thus we have the following procedure for finding a pose which for a given boundary point of X contributes to the oriented envelope:

Algorithm 3 Suppose a pose cloud $\alpha^1, \dots, \alpha^r$ and a boundary point $x \in \partial X$ with an outward normal vector n are given.

1. Let $N = 0$ and choose an index i_0 with $1 \le i_0 \le r$.
2. Find $i_{\max}$ such that $i \mapsto \langle (\alpha^{i_N})^{-1}\alpha^i \cdot x, n \rangle$ attains its maximum for $i = i_{\max}$.
3. Let $i_{N+1} = i_{\max}$ and increment N. Terminate if the sequence of i_N's becomes constant or periodic, otherwise start again with 2. $\diamond$

Alg. 3 can be used as a substitute for Alg. 2 in all later algorithms (Alg. 4 and Alg. 5). It is an entire order of magnitude faster and numerical experience shows that it indeed finds the same indices as Alg. 2.

4.4. Making the result unique

As the purpose of Alg. 2 and Alg. 3 is to compute, for a given point $x \in \partial X$, a pose α such that $\alpha \cdot x$ is a boundary point of the swept volume, it is not difficult to decide which of the k candidates suggested by Alg. 2 or Alg. 3 is the right one:

Algorithm 4 Suppose a pose cloud $\alpha^1, \dots, \alpha^r$ and vectors $n, \bar{n} = x \times n$ are given. We want to compute a pose α^i where $(\bar{n}, n)$ is an outward normal vector of $\mathscr{A}$.

1. Compute indices $i_{N-k}, \dots, i_{N-1}$ with Alg. 2 or Alg. 3.
2. Compute a mean normal vector of the bodies $\alpha^{i_{N-j}}(X)$ in the points $\alpha^{i_{N-j}} \cdot x$ by letting $n^{\text{mean}} = \sum_{j=1}^{k} A^{i_{N-j}} n$.
3. Choose $i \in \{i_{N-k}, \dots, i_{N-1}\}$ such that $\langle \alpha^i \cdot x, n^{\text{mean}} \rangle$ is maximal, i.e., x is moved as far as possible in direction n^{mean}. $\diamond$

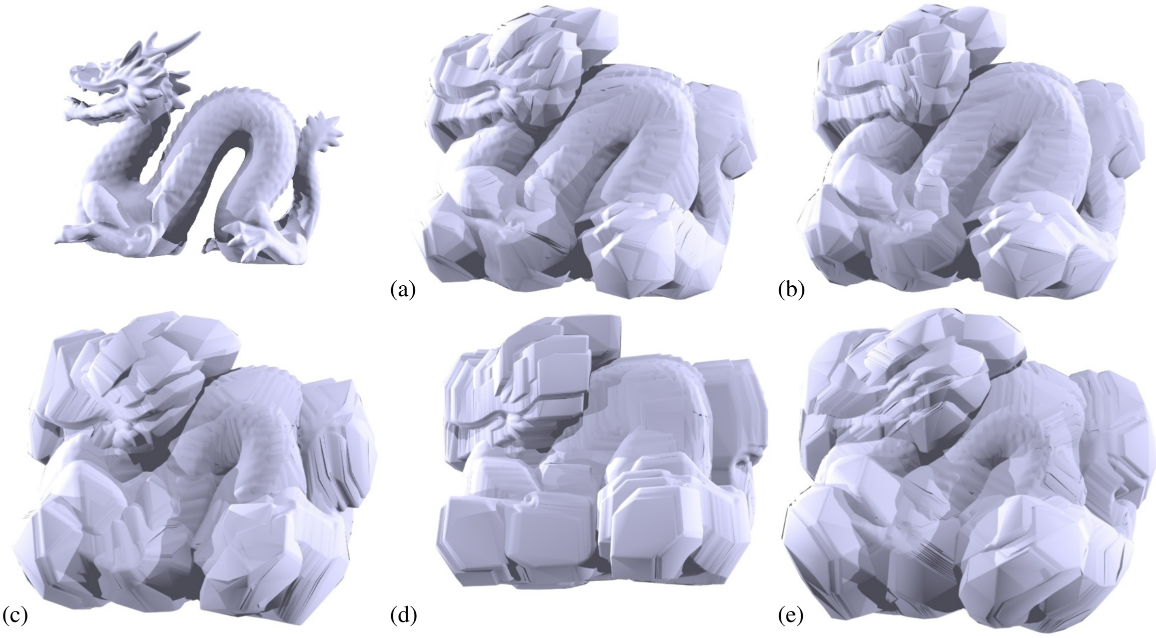


Figure 5: *Swept volumes of the Stanford dragon corresponding to cases (a)–(e) of Sec. 6.1.*

The following algorithm computes a discrete version of the oriented envelope of a triangulated rigid body X with respect to a pose cloud $\mathscr{A}$.

Algorithm 5 Suppose that ∂X is given as a triangle mesh with vertices x^j and outward unit normal vectors n^j. Further, a pose cloud $\mathscr{A}$ is given. For all x^j, use Alg. 4 to compute an index $i(j)$ from x^j, n^j and $\mathscr{A}$. Then the point $\alpha^{i(j)} \cdot x^j$ is a vertex of the oriented envelope of X with respect to $\mathscr{A}$. The connectivity of the triangulation of the oriented envelope is the same as the one of ∂X. $\diamondsuit$

According to Prop. 5, the outer part of the oriented envelope equals the boundary of the swept volume. A tame example, where the swept volume is bounded by the oriented envelope is illustrated in Fig. 4.

5. Trimming and smoothing.

The result of the algorithms above usually has self-intersections, especially if the rigid body X we started with is not convex (cf. Fig. 7). Fortunately computing the outer part of a surface is a built in feature of various software packages, and we will not consider that problem here.

Another topic is smoothness of the swept volume's boundary. High-dimensional point clouds must have much more points than three-dimensional ones if they are to represent a smooth object faithfully. We cannot expect that pose clouds have this property. Numerical experience shows that smoothing $\partial\mathscr{A}(X)$ is often necessary. In the spirit of tolerance analysis, we must not make $\mathscr{A}(X)$ smaller by smoothing, so we suggest the simple procedure below. It depends on the fact that the normal vectors in a boundary point $Ax+a$ of the swept volume is given by An, if n is the normal vector of X at x:

Algorithm 6 Assume a triangle mesh with vertices y_i and normal vectors $\widetilde{n}_i$ in the vertices.

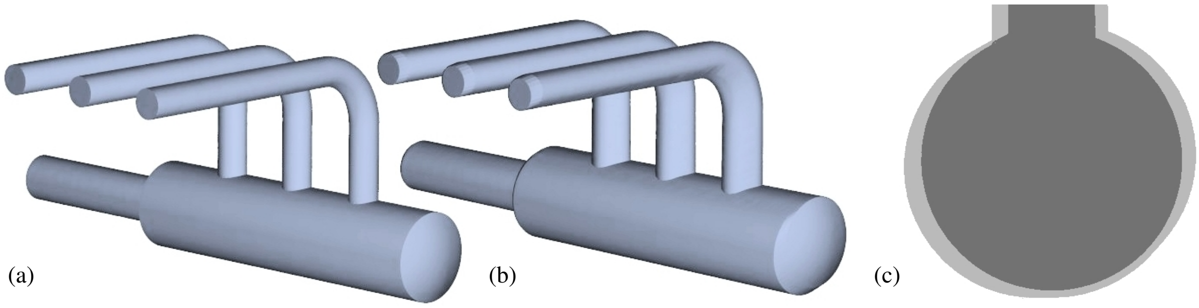
1. For all i store the neighbours of the vertex y_i in the set C_i.
2. Consider the forces $F_i = \sum_{j \in C_i} \frac{y_j - y_i}{\|y_j - y_i\|}$ exerted on y_i from its neighbours.
3. Vertices y_i where $\langle F_i, \widetilde{n}_i \rangle > 0$ are moved into an equilibrium position: Consider F_i as a function of y_i and choose d such that $\langle F_i(y_i + d\widetilde{n}_i), \widetilde{n}_i \rangle = 0$. Move y_i to $y_i + d\widetilde{n}_i$. $\diamondsuit$

Sharp edges in the data

Rounding off sharp edges in a triangulated data set X with tubular surfaces of very small radius or even zero radius has the effect that the normal vector does not abruptly change from one face to the next. This procedure has been applied to the examples of Sec. 6.2.

6. Numerical examples.

We experienced computation times of 10^{-5} seconds per vertex and pose in computing the oriented envelope, without trimming and smoothing. Depending on the size of the pose cloud, up to 7 % of points with non-unique index in Alg. 2 and Alg. 3 were observed.



(a) (b) (c)

Figure 6: *(a) Car part (courtesy AVL List GmbH). (b) Swept volume for a pose cloud representing vibration. (c) Difference between original (dark) and swept volume (light).*

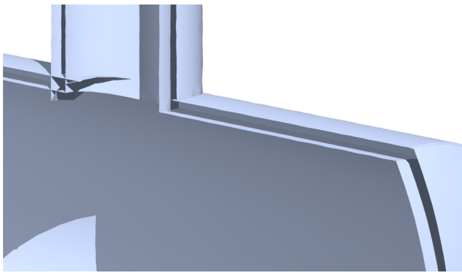


Figure 7: *Part surface (cf. Fig. 6) and the oriented envelope. Here trimming is necessary.*

6.1. Pose clouds of varying smoothness

Fig. 5 shows $\partial\mathscr{A}(X)$ where X is the well known Stanford dragon, and the pose cloud $\mathscr{A} = \alpha^1, \dots, \alpha^r$ is chosen such that $\alpha^i = \exp(d^i, \bar{d}^i)$ is as follows: In cases (a)–(c) we let $r = 200$ and choose $(d^i, \bar{d}^i)$ randomly such that (a) $\|d^i\|^2 + \|\bar{d}^i\|^2 \le 0.2$, or (b) $0.1 \le \|d^i\|^2 + \|\bar{d}^i\|^2 \le 0.2$, or (c) $\|d^i\| \le 0.2$, $\|\bar{d}^i\| \le 0.2$. In case (d) we let $r = 2^6$ and take $(d^i, \bar{d}^i)$ as the vertices of the cube $0.2 \cdot [0,1]^6$. In case (e), we let $r = 20^2$ and choose both d^i and $\bar{d}^i$ as one of 20 evenly distributed points on a sphere of radius 0.2 in $\mathbb{R}^3$.

6.2. Swept volumes of vibrating parts.

Fig. 6.a shows the evenly sampled surface of a car part X, which assumes all poses in some cloud $\mathscr{A}$. The motion of the part, i.e., the poses in $\mathscr{A}$, could for example be given by simulating vibration. The result of the action of $\mathscr{A}$ on X is shown in Fig. 6.b and Fig. 6.c. A detail of the oriented envelope is shown in Fig. 7 (the pose cloud $\mathscr{A}$ used in the figure does not come from an actual simulation).

7. Conclusion

We have shown how to compute the swept volume of a solid given by a triangle mesh under the action of a full-dimensional set of poses, which can be thought of either as tolerance zone of an imprecisely defined pose, or as a set of poses obtained by measurements or simulation. The algorithms are based on geometric properties of normal vectors of pose clouds and oriented envelopes. Thus the problem which a priori is difficult and requires searching in high dimensions, is reduced to dimension two.

Appendix: The matrix exponential and logarithm.

The matrix exponential function is defined by the power series $\exp(M) = \sum_{k\ge 0} M^k/k!$, and "log" is its local inverse such that $\log(E) = 0$. It is well known from linear algebra that for all poses $\alpha = (A, a)$ there is a pose β with

$$\beta\alpha\beta^{-1} = \begin{bmatrix} 1 & 0 \\ \tilde{a} & \tilde{A} \end{bmatrix}, \text{ where } \tilde{A} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \tilde{a} = \begin{bmatrix} 0 \\ 0 \\ a_3 \end{bmatrix}.$$

Together with $\log(\beta^{-1}\alpha\beta) = \beta^{-1}(\log\alpha)\beta$ we can now compute $\log\alpha$ for any pose α:

$$\log \begin{bmatrix} 1 & 0 \\ \tilde{a} & \tilde{A} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \tilde{a} & \tilde{S} \end{bmatrix}, \text{ with } \tilde{S} = \begin{bmatrix} 0 & -\phi & 0 \\ \phi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

"log" is only locally unique, just as the arcsin and arccos functions. Obviously "log" can be unambiguously defined in the neighbourhood of the identity defined by $-\pi < \phi < \pi$.

As to the difference between a matrix and its exponential expressed by (11), we note that the Frobenius norm $\|M\| = (\mathrm{tr}(M^T M))^{1/2}$ is multiplicative in the sense that $\|M \cdot N\| \le \|M\| \cdot \|N\|$ for all M and N. The power series of the exponential function then yields the following easy estimate: $\|\exp(M) - (E_n + M)\| = \|\sum_{k\ge 2} M^k/k!\| \le \sum_{k\ge 2} \|M\|^k/k! = g(\|M\|)$, where $g(t) = e^t - 1 - t$. For the special case of block matrices for infinitesimal motions as in (5), the previous inequality leads to (11).

Appendix: Every normal vector actually occurs.

The proof of Prop. 6 uses the concept of Brouwer degree of a mapping, its homotopy invariance and the following facts:

the degree of a diffeomorphism equals ± 1, and the degree of a mapping which is not onto equals zero [Mil65].

Proof: (of Prop. 6) Normal vectors of $\mathscr{A}$ do not change if we multiply $\mathscr{A}$ with a pose β from the left. Thus we can without loss of generality assume that $\alpha = (E, 0)$ and $\log_\alpha = \log$. We consider the mapping ν_0 which assigns to a pose its outward unit normal vector. It is well known that there is a smooth isotopy of $\log(\partial\mathscr{A})$ to a sphere, which without loss of generality can be made arbitrarily small and close to $(0,0) \in \mathbb{R}^{3+3}$. By applying "exp" we get a smooth isotopy from $\partial\mathscr{A}$ to a surface M_1, which is the exponential of a small sphere. With (11) the normal vectors of M_1 are arbitrarily close to the normal vectors of a sphere, so the mapping ν_1 which assigns to each pose in M_1 the outward unit normal vector is 1-1 and onto. It follows that $\deg(\nu_1) = \deg(\nu_0) = \pm 1$, so ν_0 is onto. $\square$

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