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Simulation of High-Resolution Granular Media

Iván Alduán[1,2], Angel Tena[2], and Miguel A. Otaduy[1]

[1]URJC Madrid, Spain
[2]NextLimit Technologies, Spain

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Abstract

Granular materials enjoy vivid motion phenomena that make them highly visually attractive. However, simulating each and every physical grain imposes an extremely high computational cost, due to the stringent resolution requirements. In this paper, we introduce a method for simulating granular media that achieves high visual resolution and high mechanical fidelity at a lower computational cost than earlier methods. Our method is based on a novel spatial decomposition of the computation of internal and external forces. The method is also highly parallelizable and configurable, allowing the artist to simulate a large range of granular materials.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Physically based modeling—

1. Introduction

Many of the materials around us, such as soil, gravel, sand, or even beans and pop corn, can be classified as granular materials. The description and simulation of their macroscopic mechanical behavior has received large attention in the civil engineering field for the study of terrains or machinery optimization, but also in computer animation [SOH99, ON03, ABC*07, LD09].

Computational models for computer animation of granular materials should be highly configurable, to bring all the power to the artist in the visual effects industry, but they should also be massively parallelizable, in order to make the most of multi-core architectures. To date, the main challenge in the simulation of granular materials is to achieve rich visual behavior and appropriate mechanical behavior without having to simulate each physical grain.

Figure 1: *Our algorithm can produce very high resolution simulations with less computational cost. In the example, over 1.6 million grains of sand fall over high-res geometry.*

The main contribution of our work lies on the simulation of internal and external forces of granular materials at two different scales. Internal granular forces are computationally demanding and require very fine temporal simulation scale. External forces, such as gravity and interaction with other objects, require, on the other hand, fine spatial simulation scale, in order to appropriately resolve contact with the environment. In this paper, we present a novel algorithm where the computationally expensive internal granular forces are simulated at a spatially large scale, while the less expensive external forces are simulated at a spatially fine scale. Overall, we achieve a good balance between spatial resolution, mechanical behavior, and computational cost.

For the simulation of internal granular forces, we build on

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the particle-based approach of Bell et al. [BYM05], but without their resolution restrictions (See Section 3). We compute internal forces on coarsely sampled *low-res* guide particles, and we then interpolate their behavior to densely sampled *high-res* particles (See Section 4). Last, we incorporate external forces to the high-res particles.

Thanks to our spatial decomposition of internal and external forces, we can simulate very high resolution scenes (over 1.6 million particles in the example in Fig. 1), while retaining high-fidelity behavior under pile formation, avalanches, or splashes. Our method is highly configurable and allows the simulation of granular media ranging from sand to pop corn. The method is also highly parallelizable, and amenable to modern multi-core architectures, as demonstrated in our evaluation examples.

2. Related Work

Granular materials have been a subject of extensive research in computational physics, and also in computer graphics. In the computational physics field, simulation methods can largely be divided into continuum-based and discrete methods. Our contribution builds on the family of discrete methods; therefore, we will focus our discussion here on discrete methods plus other particular approaches used in computer graphics.

Discrete methods concentrate the mass of the granular medium in particles or grains, and model the macroscopic behavior of the granular medium based on the contact interactions among those particles. In computational physics, discrete methods are typically separated into *event-driven* and *molecular dynamics* methods. Event-driven methods (also called hard-sphere algorithms) assume that collisions among particles happen at discrete events in time, and model the dynamics by modifying particle states at collisions. They model the particles as spheres or polyhedra. Event-driven methods are efficient when collisions take place only between pairs of particles and are separated by relatively long intervals, but they are not appropriate for phenomena with persistent contact. Some of the methods for solving rigid-body contact in computer graphics apply impulses at discrete events [MC95], and can be considered as an extension of particle-based event-driven methods. Such methods have been extended to handle simultaneous contact among many rigid bodies [GBF03], but they scale poorly with the number of bodies and are not well-suited for particles with small volume.

Molecular dynamics (also referred to as soft-sphere based algorithms or discrete-element method) was first applied by Cundall and Strack in the context of granular materials to study the motion of rock masses [CS79]. Each contact is transformed into a repulsion force based on particle interpenetration. Lee and Herrmann [LH93] added a term of static friction to this model. Non-spherical approaches to calculate the static force were investigated by Poeschel and Buchholtz [PB93, BP95], first with particles composed of spheres connected with hard springs, and later with polyhedra. Bell et al. [BYM05] model each particle as a compound of spheres and give it a rigid body behavior with torques and rotational motion taken into account. Our approach follows this line of methods. Luciani et al. [LHM95] have applied the concept of multiscale simulation to granular media.

Granular media have also been simulated using other methods in computer graphics. Height-field-based techniques are extremely efficient but they are very limited in terms of the granular phenomena they can simulate. Sumner et al. [SOH99] took a height-field approach with simple displacement and erosion rules to model footprints or tracks, and Onoue and Nishita [ON03] extended this method to multi-valued height-fields, allowing for some 3D effects. Pla et al. [PCGFM06] have applied cellular automata to model granular terrains very efficiently. Zhu and Bridson [ZB05] simulated sand using a fluid simulation framework, but their approach does not correctly handle some of the large-scale granular phenomena. Their technique has been recently applied by Lenaerts and Dutre [LD09] in the context of smoothed-particle hydrodynamics to capture the interaction between granular media and fluids. These last two methods fall in the category of continuum methods. It is important to note that our spatial force decomposition approach, although initially designed for particle-based discrete methods, may also be applicable in the context of continuum-based smoothed-particle hydrodynamics methods. Continuum models are particularly interesting because they enable modeling very large volumes of granular material, such as a fine-sand beach, in a more efficient way. Avalanches [AT01], landslides [QPHFM04], stationary shear flows [CJ04] and other large-scale phenomena can be simulated using continuum techniques.

3. Simulation of Granular Behavior

In this section, we describe the computation of internal forces of the granular material, i.e., forces due to contact and friction among grains of material. We compute such internal forces at a macroscopic scale, sampling the granular material with *low-res* (LR) particles at a rather coarse spatial resolution. In the rest of the section we give a detailed description of the dynamics model of the LR guide particles, the contact force model, and the computation of proximity queries.

3.1. Rigid Composite Particles

As discussed earlier, there are two basic ways of modeling granular particles: either as spherical or non-spherical particles. Spherical particles have shown limitations inherent to their own geometry, such as the inability to properly model the maximum angle of repose in piles, or other phenomena induced by static friction [LH93]. Non-spherical particles

Figure 2: *In a microscopic view of sand grains, we can see the irregular shapes that conform a granular medium.*

were investigated as a way to solve those problems. Fig. 2 shows a close-up of sand grains, whose irregular shape has a large influence on their macroscopic behavior.

We follow the approach of Bell et al. [BYM05] for modeling our LR guide particles. Each particle is modeled as a rigid body composed of several spherical components, hence we refer to them as *rigid composite particles*. Fig. 3 shows several possibilities for modeling the composite particles using spherical sub-particles. Other options for modeling non-spherical particles include spheres connected with hard springs, but not entirely rigid [PB93], or polyhedral shapes [BP95].

Each composite particle follows the laws of rigid body dynamics. We refer the reader to the notes of Baraff and Witkin [WB01] for a detailed description of the dynamics equations and their numerical integration. In particular, we have used an explicit Euler method for integrating velocities. The simulation of internal granular forces requires small time steps in order to properly capture the inter-particle interaction, and Euler's method proved to be sufficiently accurate at these time steps. In fact, the size of the time step is limited by the spatial resolution and the size of the particles, in order to avoid particles from entirely crossing each other. With the use of coarsely sampled LR guide particles, we can actually alleviate the time-step limitations (See a detailed discussion in Section 5).

At each time step, we first compute external and internal force *and* torque on each spherical particle, sum them up to the composite particle, and then integrate the translational and rotational motion forward. The computation of both internal and external forces, as well as the numerical integration of rigid body dynamics, can be easily parallelized over a large number of cores. Such computational parallelism was a major priority in our design.

The external forces include gravity, wind forces, and interaction with the environment (both static walls as well as dynamic rigid bodies), and are handled using state-of-the-art methods. Internal forces, on the other hand, capture the contact among grains. We model internal forces by computing contact and friction forces among LR guide particles, as we describe next.

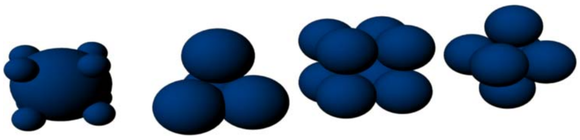

Figure 3: *Different models of rigid composite particles tested in our simulations. From left to right, composite particles formed with nine, four, eight or six spheres.*

3.2. Collision Detection

In order to calculate internal forces between particles, we need to find those particles that are potentially in contact. We enclose each composite particle in a bounding sphere. First, we test the bounding spheres against each other. Then, if the bounding spheres of two composite particles collide, we test the collision between their inner sub-particles. We use spheres as sub-particles in order to accelerate pairwise collision tests. If two sub-particles collide, we compute contact force and torque (w.r.t. the center of mass of the composite particle) using the force model described in the next subsection.

Testing all pairwise collisions between composite particles would be highly inefficient, hence we accelerate the search for potentially colliding composite particles by inserting their bounding spheres in a spatial voxelization. We set the cell size as twice the radius of the bounding sphere. Then, we only need to test for collisions between spheres in the same or adjacent cells. We reset the spatial voxelization every time step, as it proved to be more efficient than performing dynamic updates.

3.3. Internal Contact Forces

When a pair of sub-particles collide, we compute a contact force (and the corresponding torque). This force is composed of a normal term $\mathbf{F}_n$ and a shear term $\mathbf{F}_s$.

The normal force models the resistance to interpenetration, and is a nonlinear penalty force of the form:

$$\mathbf{F}_n = -(k_n \delta^{3/2} + \gamma_n \dot{\delta} \delta^{1/2}) \mathbf{n}, \tag{1}$$

where δ is the interpenetration, $\dot{\delta}$ the penetration rate, $\mathbf{n}$ the contact normal, and k_n and γ_n stiffness and damping coefficients. We discuss the parameter values that we used in our examples in Section 5.

The shear force models dynamic friction, and it takes into account maximum values of dynamic Coulomb friction and viscous friction. It is applied in the direction of the tangential relative velocity and is computed as:

$$\mathbf{F}_s = -min\left(\mu \|\mathbf{F}_n\|, k_s \|\mathbf{v}_t\|\right) \frac{\mathbf{v}_t}{\|\mathbf{v}_t\|}, \tag{2}$$

with μ and k_s the Coulomb and viscous friction coefficients, and $\mathbf{v}_t$ the tangential relative velocity.

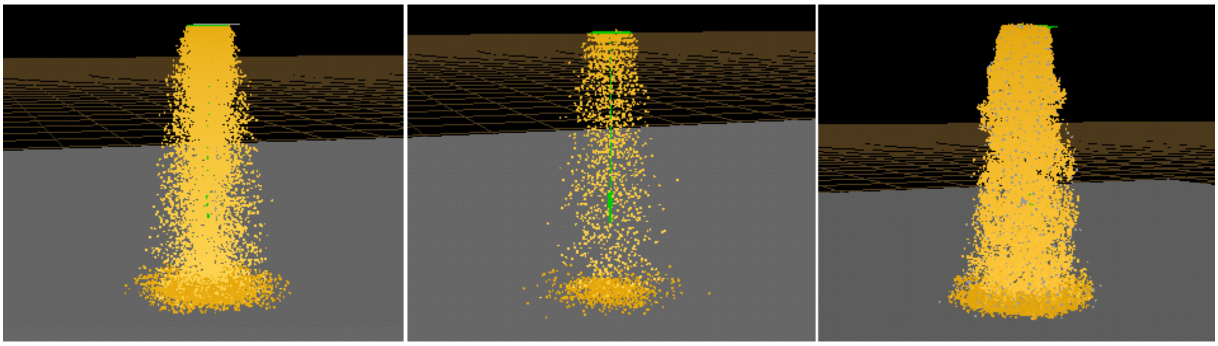

Figure 4: *Left: scene with 50000 particles simulated with full internal force computation (all particles are LR guides). Middle: scene with 2500 particles simulated with full internal force computation (all particles are LR guides). Right: scene computed with our spatial force decomposition; the internal granular behavior of the 2500 LR particles in the middle figure is interpolated to a high-resolution, and external forces are computed on 50000 HR particles. Our method obtains a speed-up of 11x compared to the scene on the left, with comparable behavior.*

The interaction between grains is dissipative due to static friction and inelastic contact. Thanks to the use of non-spherical particles, we do not need to introduce an explicit term of static friction [LH93]; static forces are captured naturally by the geometric interaction among particles. Furthermore, the use of non-spherical particles has shown better agreement with experimental results than the use of spherical particles with explicit static friction [PB93].

4. Low-Res and High-Res Particles

The coarse granular simulation described in the previous section is able to capture the qualitative macroscopic behavior of the granular medium, but the particle resolution is too low for a pleasing visual simulation. In this section, we describe how the coarse internal granular simulation is combined with a high-resolution simulation of external forces. We discuss the foundations of the force decomposition, a randomized upsampling of the particles, and the simulation of the high-resolution particles.

4.1. Motivation for the Force Decomposition

In reality, a granular medium is governed by microscopic mechanical behavior, i.e., the interactions between millions of grains. The microscopic state of the particles could then be defined by the contact forces acting on the individual particles. However, the behavior of granular flow at a macroscopic level can be qualitatively characterized even by relatively rough models with coarsely sampled particles [BKMS07]. In particular, the macroscopic behavior of a granular medium is characterized by its *relative density*, which is defined by the *void ratio* [AVS03]. Intuitively, a coarse granular simulation, such as the one described in Section 3, will maintain the qualitative behavior of a high-resolution simulation provided that the porosity and shape of the grains are maintained.

Following this observation, we decompose the forces of the granular medium. We employ coarsely sampled LR guide particles to capture the general granular behavior, and then interpolate their motion to a simulation of high-resolution (HR) particles. But, with our method, we do not simply interpolate the granular behavior. We ensure appropriate handling of boundary conditions by computing external forces (e.g., forces with the environment) on the HR particles.

The spatial decomposition of internal and external forces also allows for separate time integration. The internal forces computed on LR guide particles need to be integrated with small time steps, while external forces on the HR particles may be computed with larger time steps. A secondary yet important benefit of the force decomposition is that, thanks to the coarser and larger LR guide particles, we can increase the time step of internal forces in contrast to simulations that compute internal forces at high resolution.

4.2. Upsampling the Simulation

The space filled by the granular medium needs to be sampled with HR particles. Since the internal granular behavior of the HR particles is governed by the LR guide particles, we generate HR particles each time we add a LR guide particle to the simulation scene. Specifically, we fill the bounding sphere of a LR guide particle with a configurable but fixed number of HR particles. With regular granular flows with sufficiently close LR guides, the resulting high-resolution medium appears as a continuum flow.

We found that employing a precomputed distribution of HR particles inside the bounding sphere of each LR guide particle could easily lead to disturbing visual patterns. Instead, we have used a different uniformly-distributed random sampling for each LR particle. We have opted for the *rejection method* for uniformly sampling the interior of a sphere [Knu97]. The rejection method generates random samples

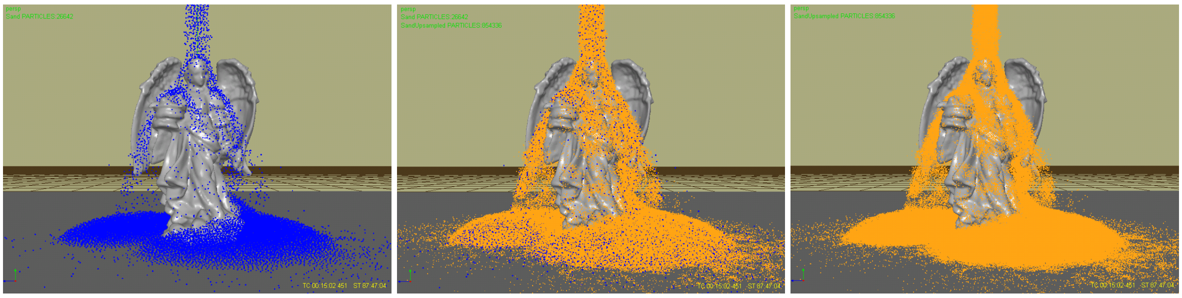

Figure 5: *Graphical representation of our spatial force decomposition. The left image shows the LR guide particles (in blue) for computing internal forces, and the right image the 32x HR particles (in yellow) for computing external forces. The middle image shows both LR and HR particles overlapped.*

inside the bounding cube by independently sampling each of the three Cartesian axes, and then keeps the random sample only if it lies inside the sphere. The sampling stops once the desired number of samples has been generated inside the sphere. We opted for the rejection method due to simplicity, but it would also be possible to directly generate a uniform distribution in the sphere.

4.3. Simulation of High-Res Particles

Our algorithm for simulating high-resolution granular particles includes two main features: (i) the interpolation of internal granular behavior, and (ii) the composition of internal and external forces.

4.3.1. Interpolation of Internal Granular Behavior

As described in Section 3, we compute the internal granular behavior on a coarse resolution. We upsample this internal granular behavior to the HR particles by interpolating the velocity field of the LR guide particles.

For each HR particle, we collect all the LR guide particles closer than an influence radius r. We interpolate the velocity field only if we collect two or more LR particles. In other words, if a HR particle is influenced by one or no LR guides, we assume that the granular medium is locally sparse and the HR particle moves freely. Then, we apply only external forces.

In case that a HR particle is influenced by two or more LR guides, we consider that the granular medium is locally dense enough, and we interpolate the internal velocity field. For each influencing LR guide, we compute a distance-based weight

$$w_i = \frac{3r - 2d_i}{4r}, \tag{3}$$

where d_i is the distance between the pair of HR and LR particles. Then, given the velocities $\mathbf{v}_i$ of the LR guide particles, we compute the interpolated velocity of the HR particle as

$$\mathbf{v} = \frac{1}{\sum w_i} \sum w_i \mathbf{v}_i. \tag{4}$$

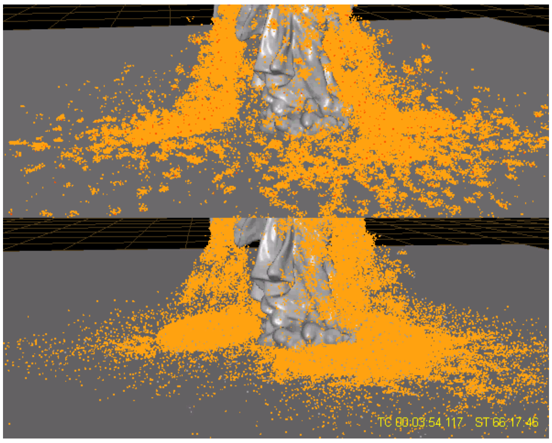

Figure 6: *Top: clumps of HR particles move rigidly with LR guide particles. Bottom: clumps are solved and sand flows appropriately thanks to a separation of dense and sparse regions in the granular medium.*

The special handling of HR particles in a sparse granular medium, i.e., with one or no influencing LR guide particles, allows us to handle correctly situations like the splash shown in Fig. 8. Without this special handling, the simulation may suffer artifacts like clumps of HR particles moving rigidly with a LR guide particle. Fig. 6 shows the improvement due to the separation of sparse and dense granular media.

4.3.2. Composition of Internal and External Forces

In our simulation, both internal and external forces are computed at the same update rate, i.e., with the same time step. Internal forces are computed only on LR guide particles, while external forces are computed both on LR and HR particles. As previously discussed, the effect of internal forces is extended from the LR guide particles to the HR particles by interpolating their velocity field.

However, we have observed that the internal granular behavior may be interpolated at a lower update rate than the force computation. In other words, we increase the computational efficiency by interpolating the velocity field of the

Figure 7: *Three frames of a simulation with over 2 million sand particles. Notice the fine sand grains and the accurate interaction with high-resolution geometry.*

LR guide particles only every N simulation time steps, with $N = \{20, 50\}$ in our examples.

5. Results and Discussion

5.1. Implementation

We have implemented the granular simulation framework in C++ as a plugin for RealFlow SDK (`www.realflow.com`) by Next Limit Technologies. This implementation required programming the particle update schemes in a modular manner and following the API of RealFlow, but it also allowed us to take advantage of its multi-core implementation. The simulations have been executed on a 2-Core, 2.13GHz CPU with 4GB of memory and an 8-core, 2.66MHz CPU, with 8GB of memory. The renders have been computed with mental ray integrated in Maya.

We have used the following set of parameter values for modeling the granular behavior (See Section 3.3). The stiffness k_n ranged from 10^4 to 10^6 N/m, depending on the type of desired material, with values ranging from 10^5 to 10^6 N/m for sand. The damping γ_n ranged from 250 to 500 N/(m/s), although we used a value of 300 in most cases. For the friction model, we used values of μ between 0.3 and 0.4, with 0.4 for sand, and a viscous friction k_s of 10^4 N/(m/s).

We have used composite LR guide particles with a radius ranging from 5 mm to 5 cm. After upsampling the simulation to model HR particles, the size of the grains is close to the size of sand grains. The size of the LR guide particles limits the size of the time step, in our case between 0.1 and 0.3 ms. Thanks to the use of coarse LR guide particles, these time steps are larger than the time steps allowed if we simulated the HR particles directly. For rendering the simulation results, we have substituted each HR particle with a small sphere of 50 polygons.

5.2. Performance Evaluation

We have compared the simulation cost of our spatial force decomposition algorithm to a standard simulation that computes internal granular force on high-resolution particles. Fig. 4-left shows a pile with 50000 particles simulated with full internal force computation. It took 16 hours to compute the 4.5-second-long simulation, with a time step of 0.1 ms. Fig. 4-center shows the same scene with 2500 particles simulated with full internal force computation. As discussed in Section 4.1, maintaining the same density ratio ensures a very similar macroscopic behavior (See the accompanying video for a dynamic comparison). Fig. 4-right shows a simulation computed with our algorithm, with 2500 LR guides and 50000 HR particles. The simulation achieves the same visual resolution as the one in Fig. 4-left, but with a time step of 0.3 ms and a total cost of 1h 25min, i.e., a speed-up of more than 11x.

5.3. Other Experiments

Our granular media simulation algorithm also has demonstrated good behavior under challenging scenarios, yet with a lower computational cost than previous methods. Fig. 8 includes snapshots from a splash of sand. The sand is initially arranged as a rocket shape, and then it splashes. A splash is a particularly challenging scenario for the upsampling and interpolation, because the granular medium transitions from a dense to a sparse distribution. As discussed in Section 4.3, our upsampling and interpolation algorithm needs to switch its procedure in this situation to avoid grains from clumping together.

Fig. 7 shows the interaction of sand with a model with complex geometry. We simulated over 40000 LR guide particle and over 1.6 million HR particles, with a time step of 0.3 ms for internal force computation, and velocity-field interpolation every 30 time steps. The complete 25-sec simulation took 69 hours to compute, at an average of 5.5 min/frame. Note that Bell et al. [BYM05] needed 26 min/frame for an avalanche with 300 thousand particles. We obtain 25x speed-up, far more than the speed-up due purely to Moore's law. In this example simulation, the spatial decomposition of forces (See Fig. 5) becomes evident. The global granular behavior is captured by the LR guide-particle simulation at a coarse scale, while the precise interaction with the high-resolution geometry is captured by computing external forces on the HR particles.

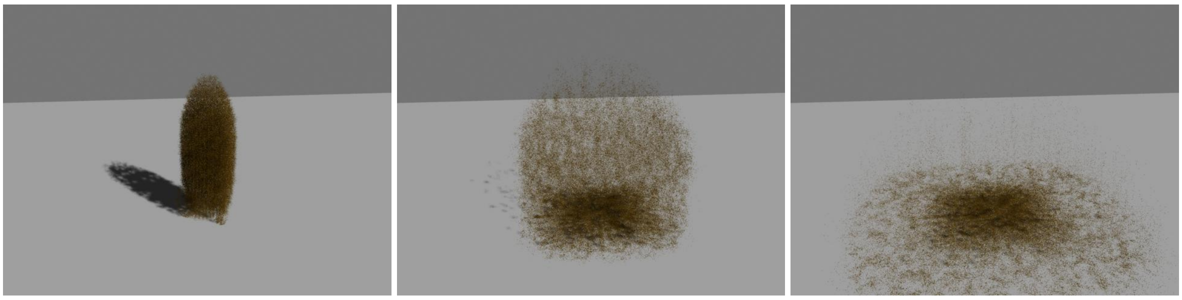

Figure 8: *A rocket of sand splashes against the ground. A splash is a particularly challenging scenario for our spatial force decomposition, as the granular medium transitions from dense to sparse distribution.*

Since our simulation algorithm is based on non-spherical rigid composite particles, it inherits their properties for accurately capturing, for example, the angle of repose in piles. By evaluating the different types of composite particles in Fig. 3, we found that the angle of repose reaches its maximum for composite particles whose shape is close to a square. The reason is that a square shape leads typically to higher porosity. We have also observed that the 9-sphere composite particle behaves closer to snowflakes or pop-corn, while the 4-sphere tetrahedral shape is best for capturing the behavior of sand. Our examples were generated with 4-sphere composite particles.

6. Limitations and Future Work

In this paper, we have presented an algorithm for simulating granular media that decomposes the computation of internal and external forces at different spatial resolutions. As a result, our algorithm does not suffer from the high computational cost of computing internal granular forces at high resolutions, and it exploits scattered-data interpolation to upsample the internal granular behavior to a visually high resolution. Our algorithm employs a robust composite-particle method [BYM05] for modeling the characteristic granular behavior, making it suitable for robustly capturing typical phenomena of granular materials like avalanches, pile formation, stick-slip motion, etc. The method is also highly parallelizable and allows the adoption of particle-based control methods [TKRP06], which makes it attractive for the special effects industry.

Nevertheless, there are many aspects that are not yet optimal. The most computationally intensive part of our method is still the calculation of internal forces, due to the rather small time steps required. As the underlying contact model relies on penalty forces and the particles are small, a small time step is needed in order to achieve a correct simulation with no instabilities or particle-crossing effects. Semi-implicit integration methods may be a possibility for increased stability and larger time steps. Time-and-space (multi-resolution) adaptive simulation [DDCB01], time-warp simulation algorithms [Mir00], and view-dependent level-of-detail techniques [ABCT07] are other possibilities for focusing the computational resources only where and when needed. Overall, although orthogonal to our research, the simulation may also be accelerated with improved nearest-neighbor search algorithms [OD08] or low-level parallelization on GPUs [YHK08].

In terms of the decomposition of internal granular behavior and external forces, there are also several possible avenues of research. The influence of microscopic granular properties and inter-granular relations on the macroscopic dynamical behavior of the whole granular material is known to be highly complicated [BKMS07]. By knowing the relationship between macroscopic and microscopic behavior of granular media, it is perhaps possible to directly add high-resolution detail in our post-process interpolation, similar to recent turbulence methods on fluids [KTJG08]. Perhaps it is also possible to blend molecular-dynamics and continuum-based or height-field methods to take advantage of both approaches, again similar to recent works on fluid [IGLF06, TRS06] or landlslide simulation [QPHFM04], or exploit level-set methods as a way to optimize our granular simulation and avoid calculations in static zones.

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