

# Magnetism and minimal surfaces – a different tool for surface design

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## Abstract

*The design of free form surfaces is usually based on NURBS and it works well to quickly get shapes that a designer intends to create. Such surfaces then have desired properties like given border lines and C1 or C2 continuity along lines where several surfaces touch. Our approach is to create surfaces with certain physical properties that designers often need. Given a closed or not closed border line, can we then find an elastic surface (comparable with a rubber surface) with the property requiring that in each point the tension is equally distributed? This is – simplified spoken – the condition for a minimal surface. Our solution does not use any differential equations but rather the following idea: We start from a patch that may be planar or part of a cylinder or any easy to define surface. This patch is tessellated in such a way that the vertices have roughly equal distances. Each point is considered to be magnetic. Now we start a converging real-time-iteration that allows the points to move according to the rules of magnetism. Border lines or parts of them may be fixed and manipulated. The corresponding algorithm is adapted from earlier algorithms by Fruchterman et al. The result is an approximation to a minimal surface that is defined by the fixed border lines. The advantage of such a surface design is twofold: First, the problem is hard to solve exactly by means of differential equations, and second the algorithm works interactively in real time. This means that the designer can change shapes almost as quickly as with conventional free form surfaces. Finally, the surface is already suitably triangulated.*

Categories and Subject Descriptors (according to ACM CCS): I.3.6 [Computer Graphics]: Minimal surfaces, Fair surface design, Force directed placement, Calculus of variations;

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## 1. Motivation

### 1.1. Motivation through natural phenomena



**Figure 1:** *Ascending air bubble, flying fox*

When air bubbles ascend in fluids (Figure 1 left), fly-

ing foxes glide with elastic skin (Figure 1 right) or flexible anemones adapt their shape in the current (Figure 2 upper right), there seems to be an infinite variety of in-between surfaces.

When water drops from a tin, each drop will change its shape dramatically in parts of milliseconds (Figure 2 left and middle left). When a stingray moves through water (Figure 2 lower right), its fins deform aesthetically and allow the fish to move efficiently over the ground.

Although not immediately obvious, there seems to be a common principle behind all those surface deformations: The surfaces of water drops, air bubbles, wing muscles etc. tend to be as tension-balanced as possible: The goal is to have more or less equally distributed tension in each surface point.

It is remarkable that people judge such surfaces as “aes-



**Figure 2:** Surfaces in nature: Water drops in different directions, anemone, wings of a stingray

thetic” or “fair”. Furthermore, it is obvious that despite the infinite manifold of the described surfaces, such surfaces are by far not arbitrarily free form surfaces – our eye seems to be trained to recognize whether the surface is “in equilibrium”. For an artist, an industrial designer or an architect this means that conventional free form surface editors are not the appropriate tool to design such surfaces.

### 1.2. Motivation through a specific question

In May 2006, Silvia Siegl came up with an interesting question (Figure 3). The artist built patterns of ceramic lenses that were spherically joined via two orthogonal rods through the lenses’ centers. The geometric analysis led to the problem of how to arrange squares with the vertices in the joints. In space, such a “lens carpet” can be deformed by knitting along the not materialized sides of the squares.



**Figure 3:** Ceramic lenses

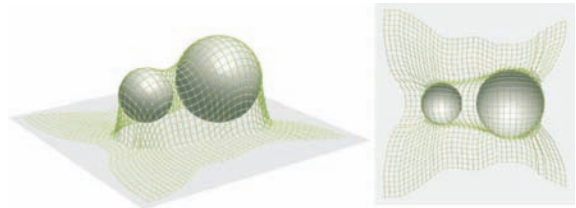
This led to a “computer solution” of the task (Figure 4). For the algorithm, the existence of the lenses is of course of no relevance.

### 1.3. First magnetic nets and their application

We can substitute the mesh by infinitely thin rods of equal lengths, joining in points that are “magnetic”. Such a “magnetic net” is movable and allows to be adapted to common



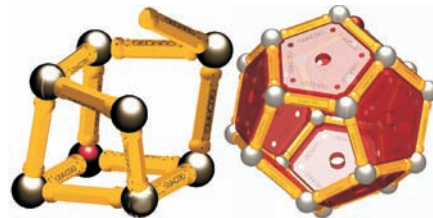
**Figure 4:** Ceramic lenses via computer



**Figure 5:** We threw a magnetic net over some obstacles

shapes. As an example, one can throw the net over some “obstacles” (Figure 5).

This is a first possibility to approximate arbitrary surfaces by means of magnetic nets. Four neighboring rods form a “skew rhomb”. Each rhomb can be split into two triangles which leads to usable triangulation of the surface – each triangle has two sides of given equal length.



**Figure 6:** Magnetic “toys” (GEOMAG) can be simulated efficiently by magnetic nets

An additional application of the concept was to simulate magnetic “toys” [Geomag] that allow to build platonic shapes and much more (Figure 6). Examples therefore can be found in [Gla07].

Figure 5 right illustrates the major disadvantage of the first type of magnetic nets: The border lines are not under control of the user. In order to avoid this drawback, the rods of the net have to be flexible and can change their lengths up to a certain extent.

Figure 7 shows more flexible nets. The different colors of the rods indicate different tensions. Figure 8 shows another application.



Figure 7: More flexible nets and some obstacles

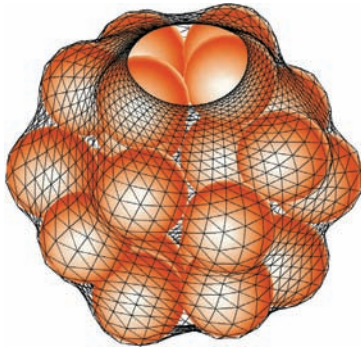


Figure 8: A “fair web” envelops 20 spheres

## 2. Variational approach for surface design

The design of surfaces by means of a variational approach – e.g. energy minimizing – is an alternative to the conventional approach via NURBS etc. [WPH07] give a good survey of the work that has been done in this direction. The design process is called “fair surface design”.

Since the satisfactory creation of surfaces with  $G^1$  continuity is very difficult, most techniques used heuristics to set e.g. extra degrees of freedom (see [Pet90] for a good overview). [MS92] have presented a method for the creation of complex smoothly shaped surfaces with  $G^1$  continuity without explicit  $G^1$  construction and could therefore circumvent artifacts like “wrinkles” which are typical of a heuristic approach. However, their method is computationally quite expensive and may therefore not be suitable for real time modification of surfaces.

In [WPH07] fair curves and polygon networks constrained to lie in surfaces or avoiding obstacles are studied. The authors define the energy of a curve network mathematically by means of cubic spline energy (the second derivative vectors of all curves are orthogonal to the surface and the first derivative vectors of incoming and outgoing curves are in equilibrium) and a tension parameter.

## 3. The Thomson problem and its generalization

In 1904, J.J. Thomson formulated the problem of how to distribute  $n$  points on a sphere in such a way that they are

“best-balanced” (Figure 9) – as part of the development of the plum pudding model of the atom ([Tho04]).

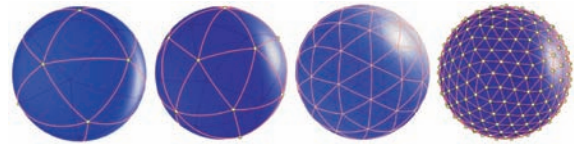


Figure 9: Thomson’s problem for 10, 15, 64 and 250 points on a sphere

A very usable solution can be found by adapting the algorithm by Fruchterman et. al. [FR91]. It belongs to the group of force directed placement (FDP) algorithms which evolved from a VLSI technique [QB79]. Eades [Ead84] based his work on the afore mentioned technique and adopted it for drawing undirected graphs. [FR91] improved the performance of Eades approach by changing the force model.

Given an undirected graph  $G = (V, E)$ , where  $V$  is a set of nodes and  $E$  is a set of edges (each edge connects two vertices  $v_n$  and  $v_m$ ,  $m \neq n$ ), Fruchterman et. al were concerned about drawing  $G$  according to some aesthetic criteria (e.g. non overlapping nodes). Their algorithm tries iteratively to improve an initial solution by displacing the nodes based on their mutual magnetic attraction and repulsion. In each iteration repulsive forces are calculated between each pair of nodes and attractive forces between nodes connected by an edge. These forces result in a disposition for each node which is applied at the end of an iteration.

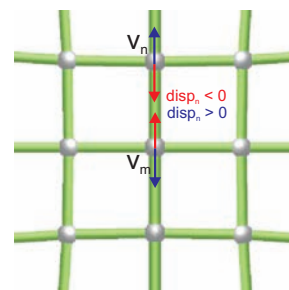


Figure 10: Graphical representation of the attractive forces

Since we are not concerned about node overlapping the calculation of the repulsive forces can be omitted which reduces the complexity from  $O(V^2)$  to  $O(E)$ . This results in a major speed improvement, which makes the algorithm suitable for real time applications. Secondly, the calculation of the attractive forces – which are responsible for the ideal length of the edges – is linear in distance instead of quadratic to improve the algorithm’s stability.

For each  $e = (v_n, v_m)$  the disposition of the node  $v_n$  is

$$disp_n = \lambda \cdot (|\vec{v}_{nm}| - v_{nm}^{ideal}) \cdot v_{nm}^0$$



where  $v_{nm}^{ideal}$  is the ideal length between node  $v_n$  and  $v_m$ . Analogously,  $disp_m = -disp_n$ . Figure 10 shows this force model.

#### 4. Minimal surfaces

So far, our magnetic nets tend to optimize the following problem: For each point of the net, the sum of all attraction vectors to the neighboring points should be minimized. This reminds immediately of a property of the minimal surfaces, where this is – roughly speaking – a substantial condition.

##### 4.1. Globally vanishing mean curvature

A minimal surface is defined in such a way that the mean curvature vanishes in any point of the surface. This includes surfaces of minimum area subject to constraints on the location of their boundary.



Figure 11: Soap films I

Physical models of area-minimizing minimal surfaces can be made by dipping a wire frame into a soap solution. Figure 11 illustrates how such a soap film satisfies the condition of tension-equalization. The same is true for elastic skins in architecture that are used to roof areas by means of tents (Figure 12).



Figure 12: Urban-Loritz-Platz, Vienna

##### 4.2. Plateau's problem

The mathematical solution to finding a surface with minimal area that spans in between given border lines was formulated and given by Plateau.

In mathematics, Plateau's problem is to show the existence of a minimal surface with a given boundary. It was formulated by J.L. Lagrange in 1760 and named after Joseph Plateau, who was interested in soap films. The problem is considered part of the calculus of variations.

In 1930, general solutions were found independently by T. Rado, whose energy minimizing method holds for *rectifiable* simple closed curves, and J. Douglas, who minimized an integral and just needs an *arbitrary* simple closed curve (the difference is thus of no practical relevance).

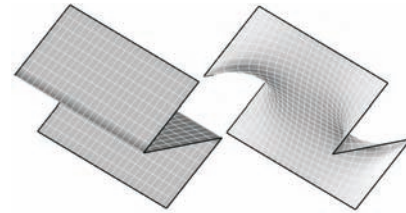


Figure 13: Minimizing energies allows to simulate minimal surfaces

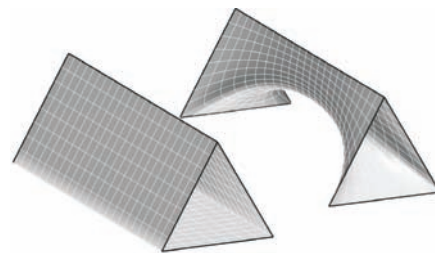


Figure 14: The boundary is a closed curve. The surface to the right has minimal area

Figure 13 and Figure 14 illustrate how our system develops such surfaces from given simple (and trivially rectifiable) closed boundaries.



Figure 15: Minimal surface in arts

Due to their aesthetics, minimal surfaces have always been of interest for artists. Figure 15 shows an installation

of Maria Wambacher at the University of Applied Arts Vienna, showing different minimal surfaces.

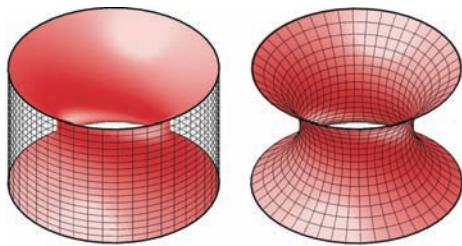
### 4.3. A possible analytical approach

[WR01] presents a possibility to efficiently find the form of the correspondent minimal surfaces (section 4) analytically: Each point of the surface is in equilibrium. Two different methods of resolution are used: The linear Force-Density Method and the non linear Dynamic Relaxation Method. The Force-Density Method needs the topology of the mesh, further the forcedensities for each element and some fixed points or borders. The result is a figure of equilibrium and a harmonically stressed shape in case of well defined forcedensities. The advantage is its linearity.

## 5. Our iterative approach for the generation of minimal surfaces

Instead of analytical solutions mentioned in the previous section, our approach works with the above described iterative algorithm derived from [FR91].

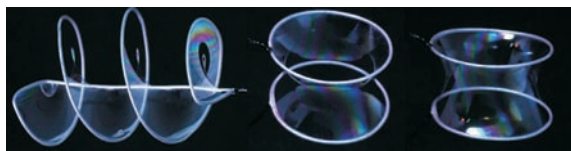
### 5.1. A topology-preserving iteration



**Figure 16:** The cylindric quadrangle-pattern converges to the topologically equivalent catenoid

A web of polygons, positioned on surfaces that are topologically equivalent to the desired surface, converges to a minimal surface, minimizing the surface energy step by step – and still in a short period of time.

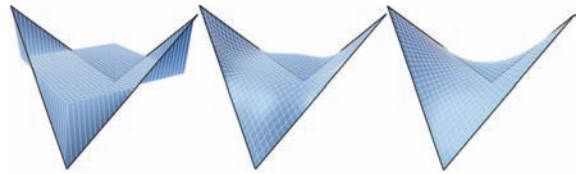
As a simple example, the cylinder in Figure 16 is topologically equivalent to the catenoid.



**Figure 17:** The helicoid and the catenoid as soap films

Figure 17 shows the experiment with soap films. In fact,

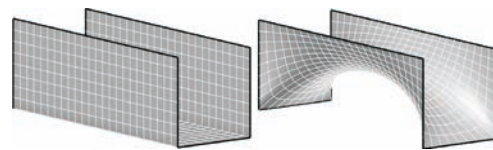
nature works very similar to our approach: Within milliseconds, the soap film has to change its shape until the surface energy is minimized. All in-between surfaces are kind of “fair surfaces” that finally deform to a more or less stable result.



**Figure 18:** Almost a hyperbolic paraboloid

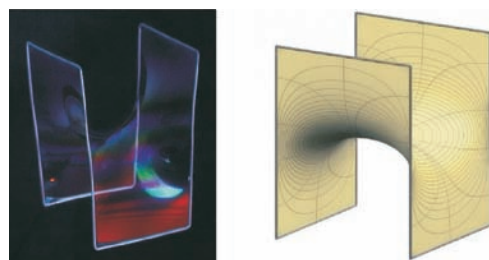
Figure 18 shows in three steps how our software proceeds the task to create a minimal surface with a skew quadrangle as boundary. The left image shows a possible topological equivalent, i.e. quadrangles in several planes. The result comes very close to a hyperbolic paraboloid, although it differs slightly from it and is well-known under the name Schwarz’ surface.

### 5.2. Intuitive triangulation



**Figure 19:** Scherk’s surface, obtained from three planar quadrangular nets

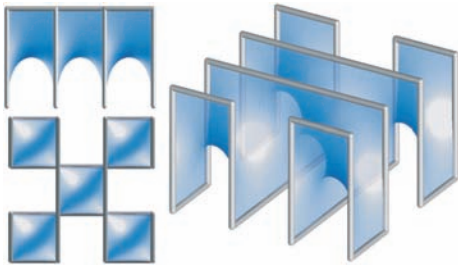
Another famous example of minimal surfaces is Scherk’s surface (Figures 19 - 21) which can in our system be easily derived from three planar quadrangular nets.



**Figure 20:** Scherk’s surface represented by a soap film and rendered by means of parametrization

This surface is typically analytically given by parameterized equations that are not very suitable for computational rendering (Figure 20 right). The surface is usually triangulated in such a way that at the borders one has very long and

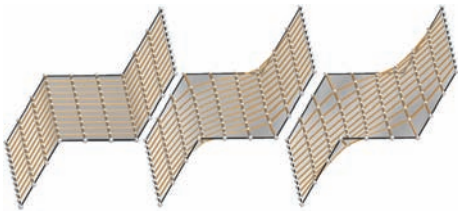
narrow triangles, no matter how fine the parametrization was done.



**Figure 21:** Adding several Scherk's surfaces leads to the famous chess-pattern surface that can be seen in Figure 21

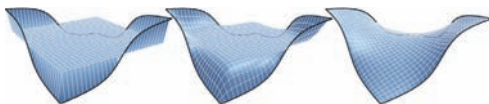
Our system creates a tessellation, where the sides of the polygons were changed in a smooth and iterative process which tries to make as little changes in the side lengths as possible. This tessellation trivially produces a comparatively smart triangulation.

**5.3. Minimal surfaces may be useful in design and "look natural"**



**Figure 22:** Designing a minimal surface that looks like a chair with elastic surface

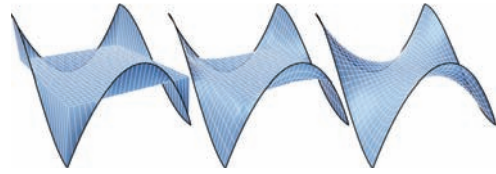
A useful application could be the design of a "chair" like in Figure 22, which of course also converges to a minimal surface. Calculation times for such surfaces stay reasonable. One can watch the deformation process in realtime. After about 30-50 frames, the result will reach a shape that is visually not distinguishable from the exact solution.



**Figure 23:** The bordering segments need of course not be straight segments

Of course, the boundary is not restricted to straight segments. When we replace such straight segments in Figure 18 by smooth curves (Figure 23, Figure 24), the algorithm will work similarly efficiently and produce a kind of minimal

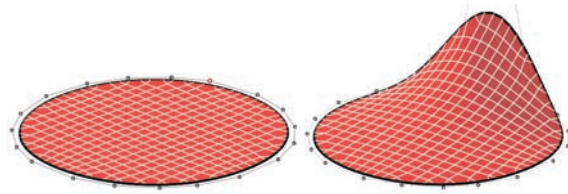
saddle-surface that reminds of the wings of a stingray (Figure 2 lower right). Such similarities may easily occur. Consider the forces that form the shape of the stingray. Very likely there is an energy-minimizing principle in behind.



**Figure 24:** Minimal surfaces are energy minimizing surfaces and probably therefore look natural

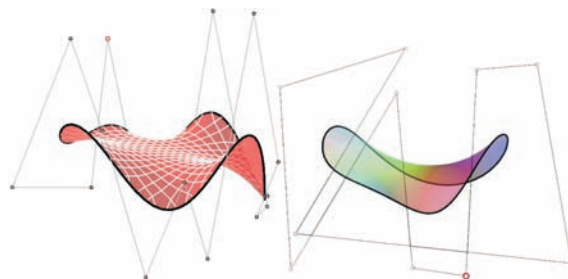
Figure 2 left shall illustrate how water reacts when equilibrium is disturbed. After a series of mostly topological equivalent in-between surfaces it returns to energy-minimizing positions.

**5.4. Topological equivalents to a circle**



**Figure 25:** Deformation of a circular pattern

Figure 25 left shows a "smartly tessellated circle". We now deform the boundary, e.g. by means of spline interpolation and – additionally – in such a way that the arc-length stays the same.

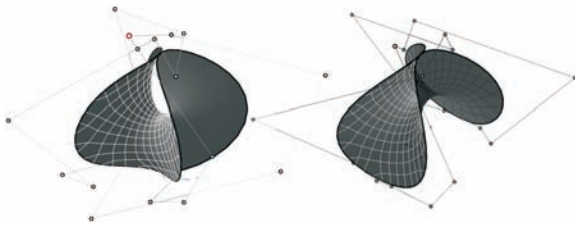


**Figure 26:** Deformation of a circular pattern

Figure 25 right and Figure 26 show possible results as the new equilibrium for the mesh.

Figure 27 illustrates that such a deformation may lead to a very famous case of minimal surface, the Enneper surface.





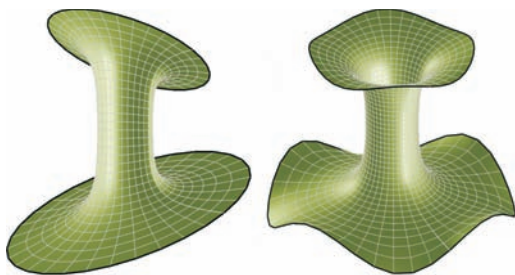
**Figure 27:** Special topological equivalent to a circular mesh: The Enneper surface



**Figure 28:** The Enneper surface reminds of molluscs

## 6. A tool for natural design

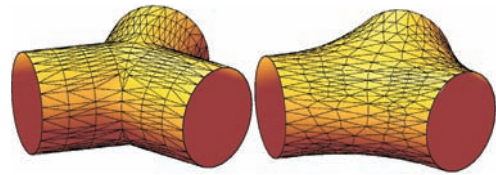
Variations of the idea of minimal topological equivalents lead to surfaces that remind of natural growth and / or smooth movements of natural surfaces (Figure 28). Figure 29 shows minimal surfaces, the boundaries of which look similar to “applications” in real world (Figure 2).



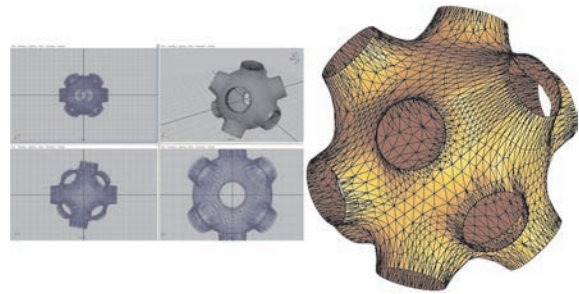
**Figure 29:** These “pseudo-catenoids” look very natural

Another example may be the topological equivalent of three intersecting cylinders as can be seen in Figure 30. It immediately reminds of natural branches (Figure 32).

Such “natural surfaces” can be created easily by first “designing something similar” by means of a professional software, second exporting the basic information into our system and finally applying the “minimizing iteration” to it. Even in-between results may be of interest (Figure 31).



**Figure 30:** Topological equivalent to three cylinders – with reasonable good triangulation



**Figure 31:** Create “something” with professional software and vary / smoothen the mesh by means of our software

Figures 32 - 34 shall give some more examples of the aesthetic output of the design system.

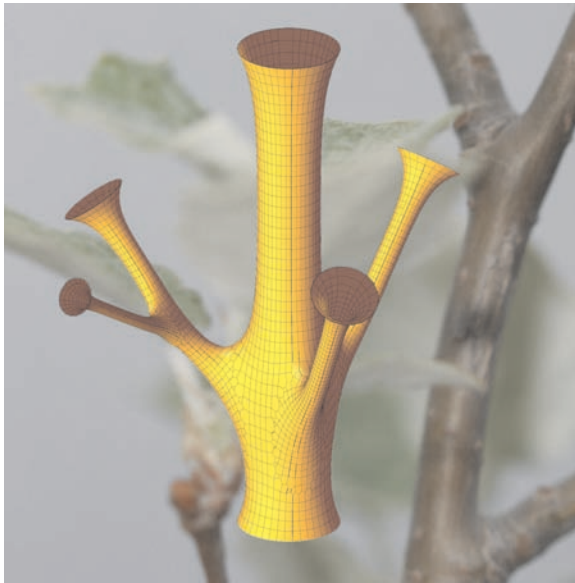
## 7. Conclusion

Minimal surfaces are a good tool to approximate elastic surfaces under tension. Our approach allows to transform given polygon-nets to topologically equivalent minimal surfaces in real time. The result is “usable triangulated”, i.e., no large or slim triangles occur. Even in-between results may be of interest when simulating physical processes like deformation of water or movements of elastic skins. Figures 32 to 34 shall demonstrate that such surfaces can be used for “natural design”.

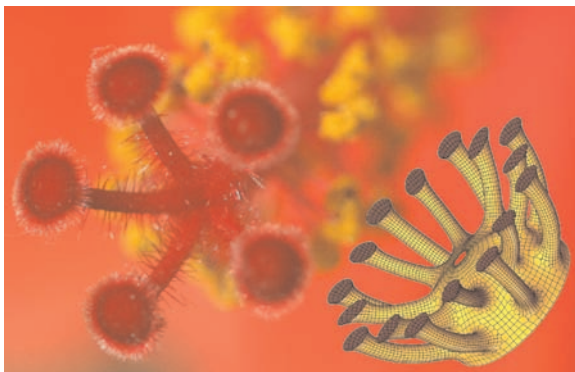
Credits: The following figures stem from [Gla07] with permission of the publisher: Figure 9, Figure 21. Photos of the soap films: Katharina Rittenschob, with permission. All other photos: G. Glaeser.

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**Figure 32:** At least some similarity, probably also caused by energy minimization



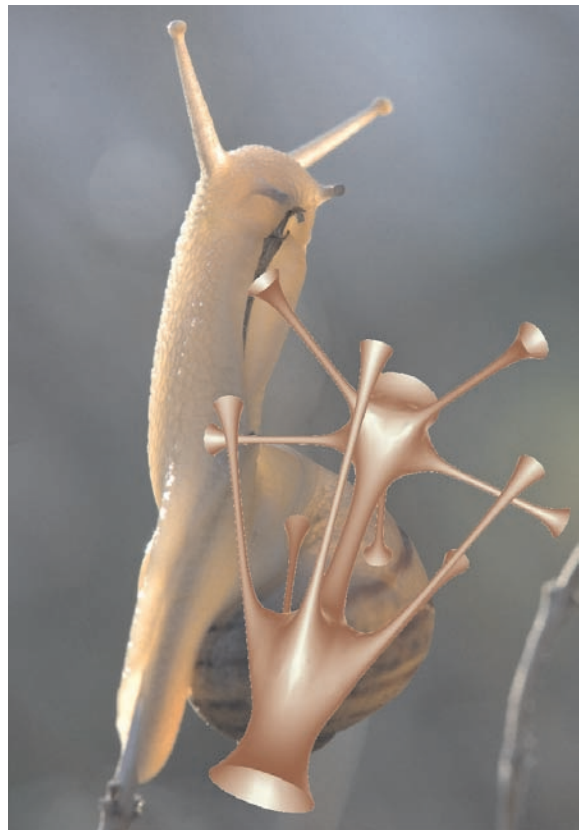
**Figure 33:** Hibiscus flower – surfaces in equilibrium

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**Figure 34:** Aesthetics and energy-optimizing surfaces

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