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Unbiased Light Transport Estimators for Inhomogeneous Participating Media

Supplementary Material: Single Scattering Algorithms

László Szirmay-Kalos[1], Iliyan Georgiev[2], Milán Magdics[1], Balázs Molnár[1], and Dávid Légrády[1]

[1] Budapest University of Technology and Economics [2] Solid Angle

Abstract

In this supplementary material we provide the details of the single scattering methods that use the medium manipulation scheme proposed in the paper entitled "Unbiased Light Transport Estimators for Inhomogeneous Participating Media".

1. Single scattering of point source lighting

In this supplementary document we consider a practically important special case of participating media rendering, the computation of single scattering of the light of a point source located at $\vec{l}$ (Figure 1). As area lights can be approximated by many point sources, and multiple scattering can be simulated by introducing *Virtual Point Lights* [ENSD12], this basic operation can be extended to a complete global illumination renderer. We take the adjoint approach and implement path tracing, thus traced particles are importons and their weight E will be the visual importance.

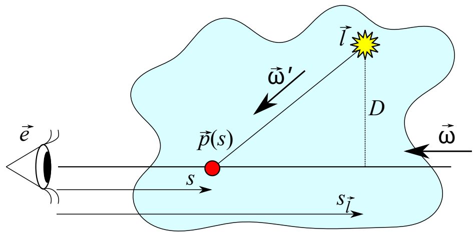

Figure 1: *Single scattering of point source lighting.*

In this special case, the radiance gathered by ray $\vec{p}(s) = \vec{e} - \vec{\omega}s$ of start $\vec{e}$, direction $\vec{\omega}$, and length S is

$$L(\vec{e},\vec{\omega}) = T_{\vec{e},\vec{\omega}}(S)L(\vec{p}(S),\vec{\omega})+$$

$$\int_0^S T_{\vec{e},\vec{\omega}}(s)\sigma_t(\vec{p}(s))a(\vec{p}(s))\rho(\vec{\omega}\cdot\vec{\omega}'(s))L^{\text{in}}(\vec{p}(s),\vec{\omega}'(s))\mathrm{d}s \quad (1)$$

where $\sigma_t(\vec{p})$ is the *extinction coefficient*, $a(\vec{p})$ is the *albedo*, $\rho(\vec{\omega},\vec{\omega}')$ is the *phase function*, $T_{\vec{e},\vec{\omega}}(s)$ is the *transmittance* between points $\vec{e}$ and $\vec{e}-\vec{\omega}s$, and $\vec{\omega}'(s)$ is the direction from light source $\vec{l}$ to point $\vec{p}(s)$. The incident radiance L^{in} due to a point source of power Φ is

$$L^{\text{in}}(s) = \frac{\Phi}{4\pi(D^2+(s-s_{\vec{l}})^2)}T_{\vec{p}(s),\vec{\omega}'(s)}(|\vec{l}-\vec{p}(s)|)$$

where D is the distance between the ray and the light source, and $s_{\vec{l}}$ is the ray parameter where the ray is the closest to the point source.

To compute the Monte Carlo quadrature of the integral of Equation 1, we need to estimate transmittance $T_{\vec{e},\vec{\omega}}(S)$, findsample points $\vec{p}(s)$ at which the scattered radiance is evaluated and multiplied by the estimate of transmittance $T_{\vec{e},\vec{\omega}}(s)$.

We estimate the transmittance as the product of the main part transmittance and the weight, i.e. the importance of a particle transmitted to the end of the considered interval, computed with the difference extinction. The particle jumps to interaction points that are generated with incrementally solving the sampling equation for given $\sigma_{\text{samp}}(\vec{p}(s))$ mimicking the absolute difference extinction. The following algorithm computes an unbiased estimate of the transmittance in interval $[0,S]$:

Transmittance($\vec{p}_{\text{start}}, \vec{\omega}, S$)
  $E = \exp(-\int_0^S \sigma_{\text{main}}(\vec{p}(\tau))\mathrm{d}\tau)$;   *// main part*
  $s = 0$;
  while ($|E| > 0$)
    // sampling equation for free flight
    Solve $\{-\log(1-\text{rand}()) = \int_0^{\Delta s}\sigma_{\text{samp}}(\vec{p}(\tau))\mathrm{d}\tau\}$ for Δs;
    $\vec{p} = \vec{p}_{\text{start}} + \vec{\omega}\Delta s$;   *// point of interaction*
    $s = s + \Delta s$;
    if ($s \geq S$) **break** ;
    $E = E(1-\sigma_{\text{diff}}(\vec{p})/\sigma_{\text{samp}}(\vec{p}))$;
    $\vec{p}_{\text{start}} = \vec{p}$;
  endwhile
  return E;
end

The key observation is that Algorithm *Transmittance* used to find $T_{\vec{e},\vec{\omega}}(S)$ generates data at interaction points from which a low variance estimator of the transmittance can be computed at an arbitrary point of the ray. This estimate is not constant between interaction points, but thanks to the main part integral, it follows an exponential fall off [JNT*11, NNDJ12]. This observation opens the possibility of using arbitrary techniques for sampling scattering points along the ray and employing the continuous transmittance function to obtain unbiased low-variance estimates. We present several techniques for finding these points, which can be used alone or even combined with the application of Multiple Importance Sampling (MIS).

1.1. Attenuation-driven sampling

Transmittance calculation produces samples, called interaction points, along the ray with density $\sigma_{\text{samp}}(\vec{p}(s))$. To prove that the sampling extinction is the density of the interaction points, let us determine the density of samples at point $\vec{p}(s)$ taking arbitrary number of jumps of length governed by the free flight sampling with sampling extinction $\sigma_{\text{samp}}(\vec{p}(s))$. We use the shorthand notation of $\sigma(s) = \sigma_{\text{samp}}(\vec{p}(s))$. The point at distance s will be reached in a single jump, i.e. with zero number of intermediate points, with pdf

$$\text{pdf}_0(s) = \sigma(s)\exp\left(-\int_0^s \sigma(\tau)\mathrm{d}\tau\right).$$

The same point can be reached in two steps jumping first to an intermediate point at distance t where $0 \le t \le s$ and then jumping from there to the point at distance s with probability density

$$\text{pdf}_1(t,s) = \sigma(t)\exp\left(-\int_0^t \sigma(\tau)\mathrm{d}\tau\right)\sigma(s)\exp\left(-\int_t^s \sigma(\tau)\mathrm{d}\tau\right) =$$

$$\sigma(t)\text{pdf}_0(s).$$

The unconditional pdf of the two step journey taking all possible intermediate t distances into account is

$$\text{pdf}_1(s) = \int_0^s \sigma(t)\mathrm{d}t \cdot \text{pdf}_0(s).$$

Now, let us examine the pdf of arriving at point of distance s via two intermediate points of distances t_1 and t_2 where $0 \le t_1 \le t_2 \le s$:

$$\text{pdf}_2(t_1,t_2,s) = \sigma(t_1)\sigma(t_2)\text{pdf}_0(s).$$

The unconditional pdf of the three step journey taking all t_1,t_2 intermediate points into account is

$$\text{pdf}_2(s) = \int_0^s \int_{t_1}^s \sigma(t_1)\sigma(t_2)\mathrm{d}t_2\mathrm{d}t_1 \cdot \text{pdf}_0(s).$$

The domain of the integral of the first factor is $0 \le t_1 \le t_2 \le s$, and its integrand $\sigma(t_1)\sigma(t_2)$ is symmetric for swapping t_1 and t_2. Thus, the integral for domain $0 \le t_1 \le t_2 \le s$ is half of the same integral for domain $t_1,t_2 \in [0,s]$:

$$\int_0^s \int_{t_1}^s \sigma(\vec{p}(t_1))\sigma(\vec{p}(t_2))\mathrm{d}t_2\mathrm{d}t_1 = \frac{1}{2}\int_0^s \int_0^s \sigma(\vec{p}(t_1))\sigma(\vec{p}(t_2))\mathrm{d}t_2\mathrm{d}t_1 =$$

$$\frac{1}{2}\left(\int_0^s \sigma(\vec{p}(t))\mathrm{d}t\right)^2$$

Thus, we obtain:

$$\text{pdf}_2(s) = \frac{1}{2}\left(\int_0^s \sigma(\vec{p}(t))\mathrm{d}t\right)^2 \cdot \text{pdf}_0(s).$$

Let us now consider the general case of allowing n intermediate points between 0 and s. The integral of $\text{pdf}_n(t_1,t_2,\ldots,t_n,s)$ is a product of an integral $\sigma(t_1)\sigma(t_2)\ldots\sigma(t_n)$ for domain $0 \le t_1 \le t_2 \le \ldots \le t_n \le s$ and $\text{pdf}_0(s)$. As the integrand in the first factor is symmetric for any permutation of t_1,t_2, etc. values, its integral can be obtained as the ratio of the same integral for domain $[0,s]^n$ and $n!$:

$$\text{pdf}_n(s) = \frac{1}{n!}\left(\int_0^s \sigma(t)\mathrm{d}t\right)^n \cdot \text{pdf}_0(s).$$

The density of samples generated with arbitrary steps is

$$d_{\text{att}}(s) = \sum_{n=0}^{\infty}\text{pdf}_n(s) = \left(\sum_{n=0}^{\infty}\frac{1}{n!}\left(\int_0^s \sigma(t)\mathrm{d}t\right)^n\right)\cdot \text{pdf}_0(s) =$$

$$\exp\left(\int_0^s \sigma(\tau)\mathrm{d}\tau\right)\sigma(s)\exp\left(-\int_0^s \sigma(\tau)\mathrm{d}\tau\right) = \sigma(s).$$

Note that the density of this sampling is not normalized since a ray may have 0, 1, 2, etc. number of interaction points. This means that the sample should be weighted by $d(s)$ instead of the product of the pdf and the number of samples.

One option is to use the sample points that are generated for the transmittance function also for the locations where scattering of the light from the point source is evaluated. The sampling density mimics the difference extinction if main part separation is turned on, which means that it does not mimic any of the factors in the integral of equation 1, thus it provides poor importance sampling and its only advantage is that sample points generated for the transmittance are reused for scattering as well.

1.2. Attenuation-driven sampling with importance re-sampling

The importance sampling aspect can be improved by the concept of *Sampling Importance Re-sampling* (SIR), which randomly reuses sample $\vec{p}(s_i)$ generated during the attenuation calculation, with probability

$$\lambda\frac{T_{\vec{e},\vec{\omega}}(s_i)\rho(\vec{\omega},\vec{\omega}'(s_i))\sigma_t(\vec{p}(s_i))a(\vec{p}(s_i))}{\sigma_{\text{samp}}(\vec{p}(s_i))}$$

where λ is an appropriate factor that guarantees that these ratios are indeed probabilities, i.e. they are not greater than 1:

$$\lambda = \min_i \frac{\sigma_{\text{samp}}(\vec{p}(s_i))}{T_{\vec{e},\vec{\omega}}(s_i))\rho(\vec{\omega},\vec{\omega}'(s_i))\sigma_t(\vec{p}(s_i))a(\vec{p}(s_i))}.$$

This rejection sampling scheme increases the variance but saves the expensive shadow ray computation of those interaction points that would have negligible scattered radiance. The density of this scheme is a composition of free flight sampling with σ_{samp} and importance re-sampling:

$$d_{\text{SIR}}(s) = \lambda T_{\vec{e},\vec{\omega}}(s)P(\vec{\omega},\vec{\omega}'(s))\sigma_t(\vec{p}(s))a(\vec{p}(s)).$$

1.3. Source-driven sampling

As it has been pointed out in [KF12], factor $L^{\text{in}}(s)$ of the integral in Equation 1 can cause significant variance since $1/(D^2+(s-s_{\vec{l}})^2)$ can be unbounded and may have a singularity if the ray goes close to the source making D small. Thus, it is worth sampling proportionally to this factor, which is possible with the application of the Cauchy distribution [KF12]. This option is called *source-driven sampling*. The pdf of Cauchy distribution is

$$\text{pdf}_{\text{source}}(s) = \frac{D}{(\theta(s_{\max})-\theta(s_0))(D^2+(s-s_{\vec{l}})^2}$$

where $\theta(s) = \tan^{-1}((s-s_{\vec{l}})/D)$ is the angle corresponding to ray parameter s.

1.4. Scattering-driven sampling

The sampling process can also mimic $T_{\vec{e},\vec{\omega}}(s)\sigma_t(\vec{p}(s))$ by using a piece-wise linear approximation of the extinction coefficient to allow the analytic solution of the sampling equation. This option is called *scattering-driven sampling*. The pdf of this sampling is

$$\text{pdf}_{\text{scatter}}(s) = \tilde{\sigma}_t(\vec{p}(s)) \exp\left(-\int_0^s \tilde{\sigma}_t(\vec{p}(\tau))\mathrm{d}\tau\right)$$

where $\tilde{\sigma}_t(\vec{p}(s))$ is the piece-wise linear approximation of the extinction coefficient.

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