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Fast Simulation of Deformable Models in Contact Using Dynamic Deformation Textures

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APPENDIX: Lagrangian Motion Equations

Shabana [Sha89] describes the derivation of motion equations from general Lagrangian mechanics for the case of a deformable model with separation of rigid-body motion. In this supplementary document we summarize this derivation for the interested reader.

From Lagrangian continuum mechanics, the motion equations of a deformable body with generalized coordinate set $\mathbf{q}$ can be written in their general form as [GPS02]:

$$\frac{d}{dt}\left(\frac{\partial \mathscr{T}}{\partial \dot{\mathbf{q}}}\right)^T - \left(\frac{\partial \mathscr{T}}{\partial \mathbf{q}}\right)^T + \left(\frac{\partial \mathscr{F}}{\partial \mathbf{q}}\right)^T + \partial_{\mathbf{x}}\mathscr{E} = \bar{\mathbf{Q}}, \tag{1}$$

where $\mathscr{T}$ is the kinetic energy of the body, $\mathscr{F}$ is the work done by the body against dissipative forces, $\mathscr{E}(\mathbf{x})$ is the elastic energy of the body, and $\bar{\mathbf{Q}}$ is the vector of generalized external forces which includes gravity and contact forces.

Elastic Energy

The virtual work due to elastic forces can be written as

$$\delta W_e = -\int_V \sigma^T \delta\varepsilon dV \tag{2}$$

where ε and σ are the stress and strain vectors. With our choice of linear strain model, the strain can be written in terms of the displacement field as $\varepsilon = \mathbf{B}\mathbf{u}_e$, where $\mathbf{B}$ is a differential operator matrix. In terms of the generalized elastic coordinates of the body, this becomes $\varepsilon = \mathbf{B}\mathbf{S}\mathbf{q}_e$.

For a linear isotropic material, the constitutive relationship between stress and strain is $\sigma = \mathbf{E}\varepsilon$, with $\mathbf{E}$ the symmetric matrix of elastic coefficients, defined by the two Lamé material constants λ and μ. This enables writing the stress vector in terms of the generalized elastic coordinates. Substituting ε and σ into (2) yields an expression for the virtual work due to elastic forces:

$$\delta W_e = -\mathbf{q}_e^T\left[\int_V (\mathbf{BS})^T\mathbf{EBS}dV\right]\delta\mathbf{q}_e = -\mathbf{q}_e^T\mathbf{K}_e\delta\mathbf{q}_e \tag{3}$$

Here, $\mathbf{K}_e$ is the symmetric positive definite stiffness matrix associated with the elastic coordinates of the body. The generalized stiffness matrix $\bar{\mathbf{K}}$ can be formed from $\mathbf{K}_e$ as $\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{K}_e \end{pmatrix}$. Following equation (3), the virtual work due to elastic forces can be written as $\delta W_e = \bar{\mathbf{Q}}_e^T\delta\mathbf{q}_e$, where $\bar{\mathbf{Q}}_e = -\bar{\mathbf{K}}\mathbf{q}_e$ is regarded as a generalized force acting on the body, or equivalently, $-\partial_{\mathbf{x}}\mathscr{E}$.

Equations of Motion

The various terms of the Lagrangian equation (1) can be rewritten by integrating the kinetic energy $\mathscr{T}$, the work produced against dissipative forces $\mathscr{F}$, and the elastic energy $\mathscr{E}$ over the entire deformable body, exploiting the texture-based discretization of the deformable layer described in Section 3.2 in the paper:

$$\frac{d}{dt}\left(\frac{\partial \mathscr{T}}{\partial \dot{\mathbf{q}}}\right)^T - \left(\frac{\partial \mathscr{T}}{\partial \mathbf{q}}\right)^T = \bar{\mathbf{M}}\ddot{\mathbf{q}} + \dot{\bar{\mathbf{M}}}\dot{\mathbf{q}} - \left[\frac{\partial}{\partial \mathbf{q}}\left(\frac{1}{2}\dot{\mathbf{q}}^T\bar{\mathbf{M}}\dot{\mathbf{q}}\right)\right]^T, \tag{4}$$

$$\left(\frac{\partial \mathscr{F}}{\partial \mathbf{q}}\right)^T = \bar{\mathbf{D}}\dot{\mathbf{q}}, \tag{5}$$

$$\partial_{\mathbf{x}}\mathscr{E} = \bar{\mathbf{K}}\mathbf{q}, \tag{6}$$

where $\bar{\mathbf{M}}$, $\bar{\mathbf{D}}$, and $\bar{\mathbf{K}}$ are, respectively, the generalized mass, damping, and stiffness matrices of the deformable body. They are obtained by integration with linear elements and linear basis functions, and for their exact expressions we refer to [Sha89]. Note that, due to definition of the elastic energy based on the displacement field, the generalized elastic forces only depend on the elastic coordinates.

We define the mass matrix $\mathbf{M} = (\mathbf{P}^+)^T\bar{\mathbf{M}}\mathbf{P}^+$, damping matrix $\mathbf{D} = (\mathbf{P}^+)^T\bar{\mathbf{D}}\mathbf{P}^+$, and stiffness matrix $\mathbf{K} = (\mathbf{P}^+)^T\bar{\mathbf{K}}\mathbf{P}^+$. We also define transformed external forces $\mathbf{Q} = (\mathbf{P}^+)^T\bar{\mathbf{Q}}$, and a *quadratic velocity vector* $\mathbf{Q}_v$ as

$$\mathbf{Q}_v = (\mathbf{P}^+)^T\left(-\dot{\bar{\mathbf{M}}}\dot{\mathbf{q}} + \left[\frac{\partial}{\partial \mathbf{q}}\left(\frac{1}{2}\dot{\mathbf{q}}^T\bar{\mathbf{M}}\dot{\mathbf{q}}\right)\right]^T\right). \tag{7}$$

After some algebraic manipulation, and applying $\mathbf{v} = \mathbf{P}\dot{\mathbf{q}}$

(See Appendix A in the paper), the system of motion equations can be reduced to its familiar form:

$$\begin{cases} \mathbf{M}\dot{\mathbf{v}} = \mathbf{Q} + \mathbf{Q}_v - \mathbf{K}\mathbf{q} - \mathbf{D}\mathbf{v} = \mathbf{F}, \\ \dot{\mathbf{q}} = \mathbf{P}^{+}\mathbf{v}. \end{cases} \tag{8}$$

External Forces

Generalized forces $\bar{\mathbf{Q}}$ can be computed from world-frame forces $\mathbf{f}$ applying the principle of virtual work. Using the kinematic relationship $\dot{\mathbf{x}} = \mathbf{LPv}$ (See Appendix A in the paper), a world-frame force $\mathbf{f}_p$ applied at a point p on the deformable body induces a generalized force $\bar{\mathbf{Q}}_p = \mathbf{P}^T\mathbf{L}(p)^T\mathbf{f}_p$.

Mass Matrix

The mass matrix $\mathbf{M}$ has the following structure [Sha89]:

$$\mathbf{M} = \begin{pmatrix} m\mathbf{I}_3 & \mathbf{R}\tilde{\mathbf{S}}_t & \mathbf{R}\bar{\mathbf{S}} \\ -\tilde{\mathbf{S}}_t\mathbf{R}^T & \mathbf{I}_\theta & \mathbf{I}_{\theta e} \\ \bar{\mathbf{S}}^T\mathbf{R}^T & \mathbf{I}_{\theta e}^T & \mathbf{M}_e \end{pmatrix}, \tag{9}$$

with mass integral m, $\mathbf{I}_\theta$ the usual inertia tensor, time dependent inertia shape integrals

$$\mathbf{S}_t = \int \rho \mathbf{u} dV, \tag{10}$$

$$\bar{\mathbf{S}} = \int \rho \mathbf{S} dV, \tag{11}$$

and

$$\mathbf{I}_{\theta e} = \int \rho \tilde{\mathbf{u}} \mathbf{S} dV. \tag{12}$$

Quadratic Velocity Vector

From the mass matrix $\mathbf{M}$ and (7), the quadratic velocity vector reverts to [Sha89]:

$$\mathbf{Q}_v = \begin{bmatrix} \mathbf{Q}_{vc} \\ \mathbf{Q}_{ve} \end{bmatrix}, \qquad \begin{aligned} \mathbf{Q}_{vc} &= \begin{pmatrix} -\mathbf{R}\omega \times (\omega \times \mathbf{S}_t + 2\bar{\mathbf{S}}\mathbf{v}_e) \\ -2\mathbf{I}_\theta\omega - 2\mathbf{I}_{\theta e}\mathbf{v}_e - \dot{\mathbf{I}}_\theta\omega \end{pmatrix}, \\ \mathbf{Q}_{ve} &= \left(-\mathbf{M}_e[\tilde{\omega}^2]\underline{\mathbf{u}} - 2\mathbf{M}_e[\tilde{\omega}]\mathbf{v}_e\right), \end{aligned} \tag{13}$$

where $[\mathbf{A}]$ denotes a block diagonal matrix with $\mathbf{A}$ replicated in every block, and $\underline{\mathbf{u}}$ is a column vector that packs the body-frame position $\mathbf{u}$ of all simulation nodes.

References

[GPS02] GOLDSTEIN H., POOLE C., SAFKO J.: *Classical Mechanics, 3rd Ed.* Addison Wesley, 2002.

[Sha89] SHABANA A. A.: *Dynamics of Multibody Systems*. John Wiley and Sons, 1989.