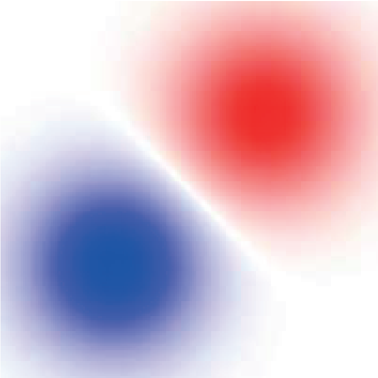
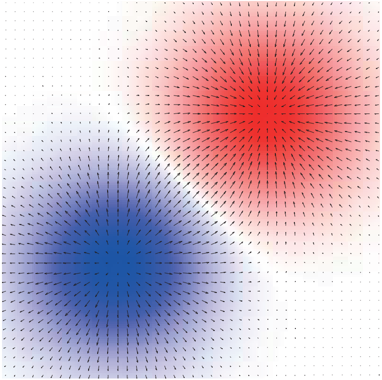
Laplacian in one minute



Smooth scalar field f

Laplacian in one minute

- Gradient $\nabla f(x)$ = 'direction of the steepest increase of f at x'



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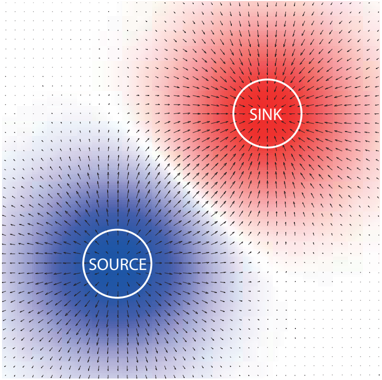
Laplacian in one minute

- Gradient $\nabla f(x) =$ 'direction of the steepest increase of f at x'
- Divergence $\mathrm{div}(F(x)) =$ 'density of an outward flux of F from an infinitesimal volume around x'



Smooth vector field F

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- Gradient $\nabla f(x) =$ 'direction of the steepest increase of f at x'
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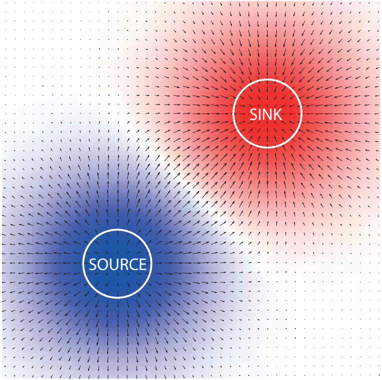
  Divergence theorem:

  $$\int_V \text{div}(\text{F}) dV = \int_{\partial V} \langle F, \hat{n} \rangle dS$$

  '$\sum$ sources + sinks = net flow'



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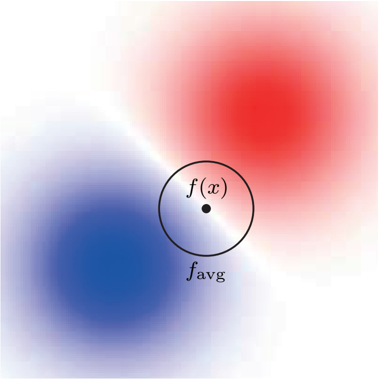


- Laplacian $\Delta f(x) = -\mathrm{div}(\nabla f(x))$

  'difference between $f(x)$ and the average of f on an infinitesimal sphere around x' (consequence of the Divergence theorem)

*We define Laplacian with negative sign

Physical application: heat equation

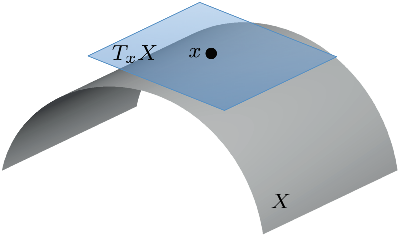
$$f_t = -c\Delta f$$

Newton's law of cooling: rate of change of the temperature of an object is proportional to the difference between its own temperature and the temperature of the surrounding

c [m^2/sec] $=$ thermal diffusivity constant (assumed $= 1$)

Riemannian geometry in one minute

- Tangent plane T_xX = local Euclidean representation of manifold (surface) X around x

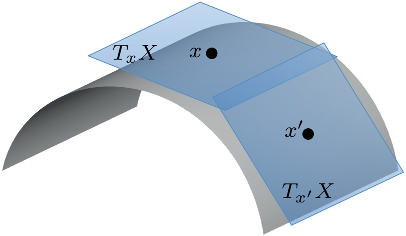


Riemannian geometry in one minute

- Tangent plane $T_x X$ = local Euclidean representation of manifold (surface) X around x
- Riemannian metric

$$\langle \cdot, \cdot \rangle_{T_x X} : T_x X \times T_x X \to \mathbb{R}$$

depending smoothly on x



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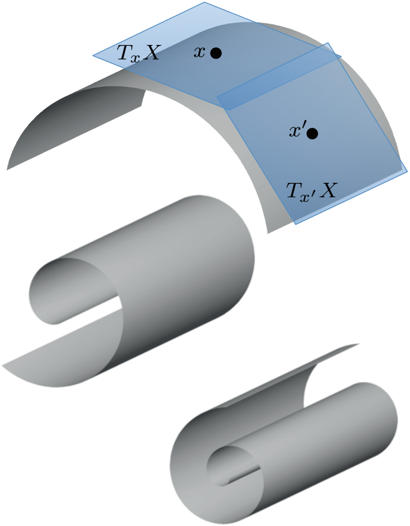

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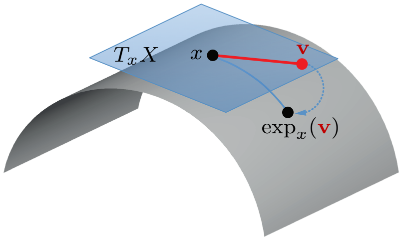
  $$\exp_x : T_xX \to X$$

  'unit step along geodesic'



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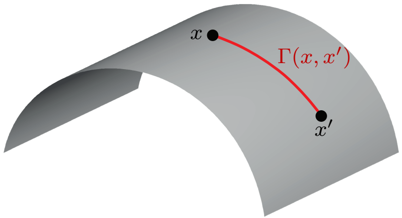
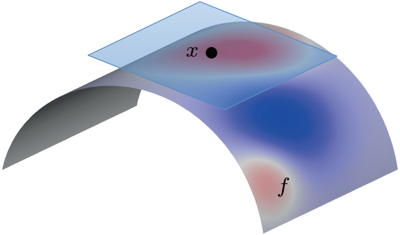
- Geodesic = shortest path on X between x and x'

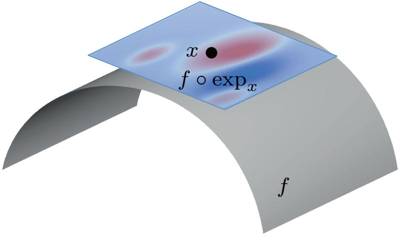


Laplace-Beltrami operator



Smooth field $f : X \to \mathbb{R}$

Laplace-Beltrami operator



Smooth field $f \circ \exp_x : T_xX \to \mathbb{R}$

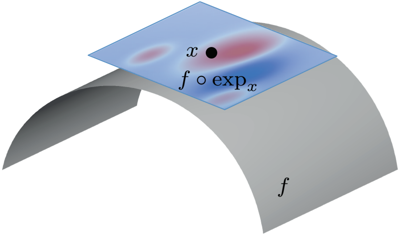
Laplace-Beltrami operator

- Intrinsic gradient

$$\nabla_X f(x) = \nabla(f \circ \exp_x)(\mathbf{0})$$

Taylor expansion

$$(f \circ \exp_x)(\mathbf{v}) \approx f(x) + \langle \nabla_X f(x), \mathbf{v} \rangle_{T_x X}$$



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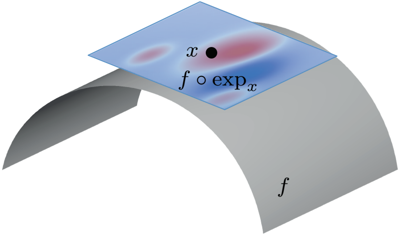

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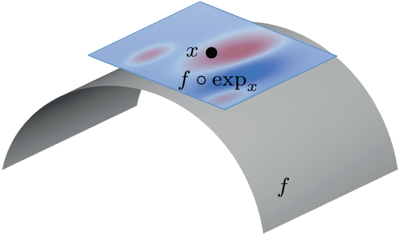

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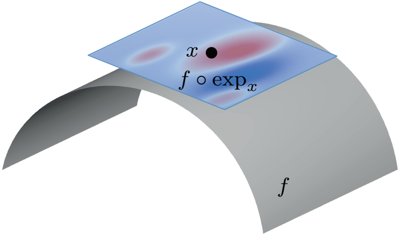

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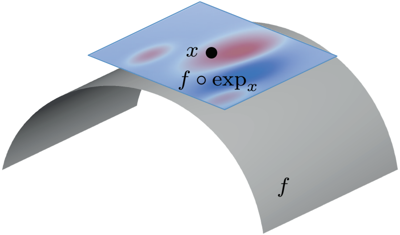
$$\Delta_X f(x) = \Delta(f \circ \exp_x)(\mathbf{0})$$

  - Intrinsic (expressed solely in terms of the Riemannian metric)
  - Isometry-invariant

Laplace-Beltrami operator



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$$\Delta_X f(x) = \Delta(f \circ \exp_x)(\mathbf{0})$$

  - Intrinsic (expressed solely in terms of the Riemannian metric)
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  - Self-adjoint $\langle \Delta_X f, g \rangle_{L^2(X)} = \langle f, \Delta_X g \rangle_{L^2(X)}$

Laplace-Beltrami operator

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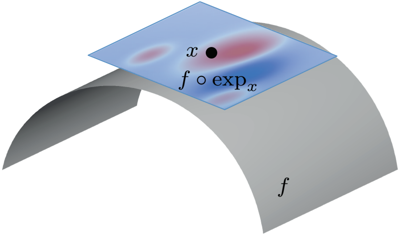

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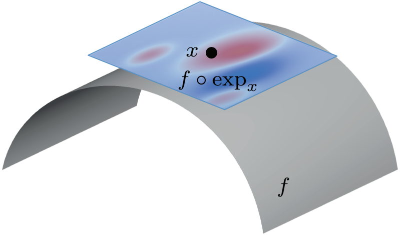

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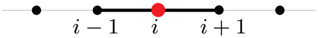
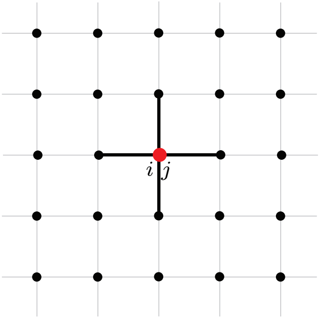
  - Positive semidefinite $\Rightarrow$ non-negative eigenvalues

Discrete Laplacian (Euclidean)



One-dimensional

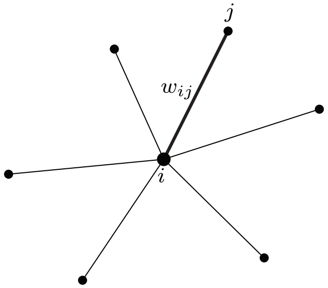
$$(\Delta f)_i \approx 2f_i - f_{i-1} - f_{i+1}$$



Two-dimensional

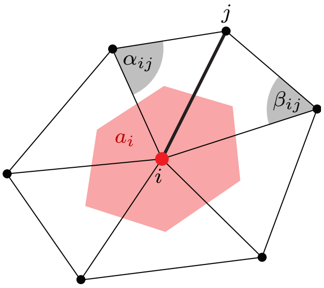
$$\begin{aligned}(\Delta f)_{ij} &\approx 4f_{ij} - f_{i-1,j} - f_{i+1,j} \\ &- f_{i,j-1} - f_{i,j+1}\end{aligned}$$

Discrete Laplacian (non-Euclidean)



Undirected graph (V, E)

$$(\Delta f)_i \approx \sum_{(i,j)\in E} w_{ij}(f_i - f_j)$$



Triangular mesh (V, E, F)

$$(\Delta f)_i \approx \frac{1}{a_i} \sum_{(i,j)\in E} \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2} (f_i - f_j)$$

$a_i =$ local area element

Tutte 1963; MacNeal 1949; Duffin 1959; Pinkall, Polthier 1993

Physical application: heat equation

$$f_t = -c\Delta f$$

Newton's law of cooling: rate of change of the temperature of an object is proportional to the difference between its own temperature and the temperature of the surrounding

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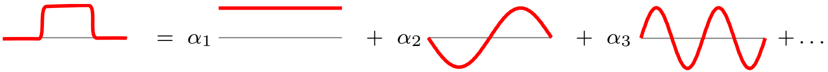
Fourier analysis (Euclidean spaces)

A function $f : [-\pi, \pi] \to \mathbb{R}$ can be written as Fourier series

$$f(x) = \sum_{\omega} \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi) e^{i\omega\xi} d\xi \quad e^{-i\omega x}$$



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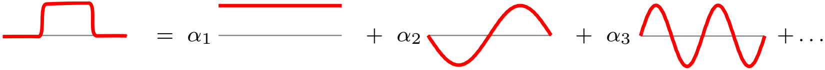

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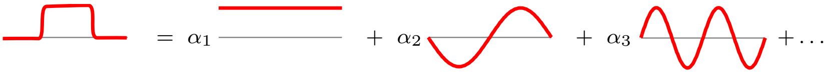
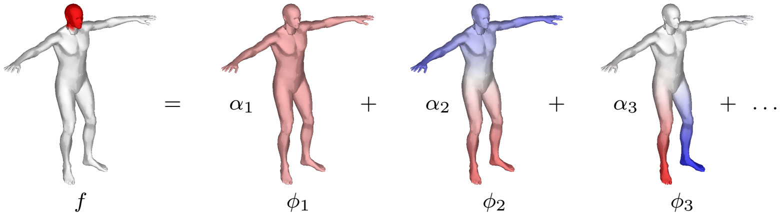


Fourier basis = Laplacian eigenfunctions: $\Delta e^{-i\omega x} = \omega^2 e^{-i\omega x}$

Fourier analysis (non-Euclidean spaces)

A function $f : X \to \mathbb{R}$ can be written as Fourier series

$$f(x) = \sum_{k \geq 1} \underbrace{\int_X f(\xi)\phi_k(\xi)d\xi}_{\hat{f}_k = \langle f, \phi_k \rangle_{L^2(X)}} \phi_k(x)$$



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Convolution (Euclidean spaces)

Given two functions $f, g : [-\pi, \pi] \to \mathbb{R}$ their convolution is a function

$$(f \star g)(x) = \int_{-\pi}^{\pi} f(\xi) g(x - \xi) d\xi$$

d'Alembert 1754

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Convolution Theorem: Fourier transform diagonalizes the convolution operator

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Convolution Theorem: Fourier transform diagonalizes the convolution operator $\Rightarrow$ convolution can be computed in the Fourier domain as

$$f \star g = \mathcal{F}^{-1}(\mathcal{F}f \cdot \mathcal{F}g)$$

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Convolution (non-Euclidean spaces)

Generalized convolution of $f, g \in L^2(X)$ can be defined by analogy

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Shuman et al. 2013

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Shuman et al. 2013; Henaff, Bruna, LeCun 2015

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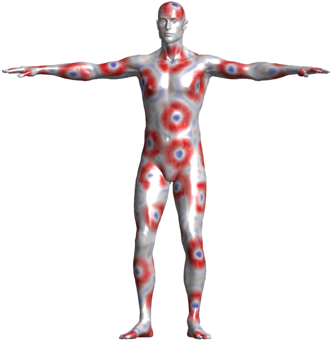
- Problem: Filter coefficients depend on basis $\{\phi_k\}_{k \geq 1}$
  $\Rightarrow$ **does not generalize to other domains!**

Shuman et al. 2013; Henaff, Bruna, LeCun 2015

Convolution (non-Euclidean spaces)

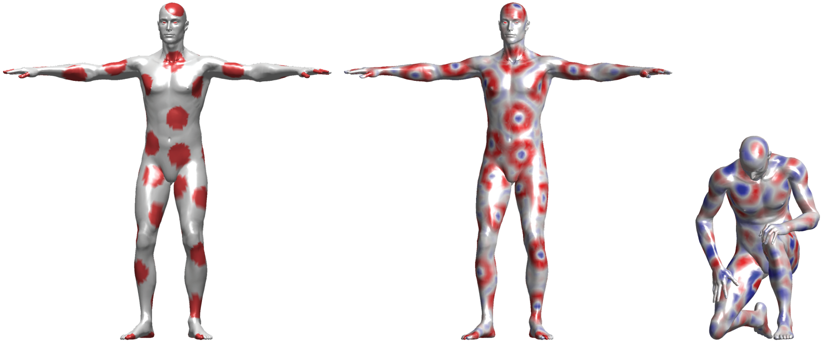
Function f



Filtered function $\tilde{f}$

Henaff, Bruna, LeCun 2015

Convolution (non-Euclidean spaces)



Function f | Filtered function f | Same filter another shape

Henaff, Bruna, LeCun 2015

Heat diffusion on manifolds

$$\begin{cases} f_t(x,t) = -\Delta_X f(x,t) \\ f(x,0) = f_0(x) \end{cases}$$

- $f(x,t)$ = amount of heat at point x at time t
- $f_0(x)$ = initial heat distribution

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Solution of the heat equation expressed through the heat operator

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$$\begin{aligned} f(x,t) &= e^{-t\Delta_X} f_0(x) = \sum_{k\geq 1} \langle f_0, \phi_k \rangle_{L^2(X)} e^{-t\lambda_k} \phi_k(x) \\ &= \int_X f_0(\xi) \underbrace{\sum_{k\geq 1} e^{-t\lambda_k} \phi_k(x)\phi_k(\xi)}_{\text{heat kernel } h_t(x,\xi)} d\xi \end{aligned}$$

Heat diffusion on manifolds

$$\begin{cases} f_t(x,t) = -\Delta_X f(x,t) \\ f(x,0) = f_0(x) \end{cases}$$

- $f(x,t)$ = amount of heat at point x at time t
- $f_0(x)$ = initial heat distribution

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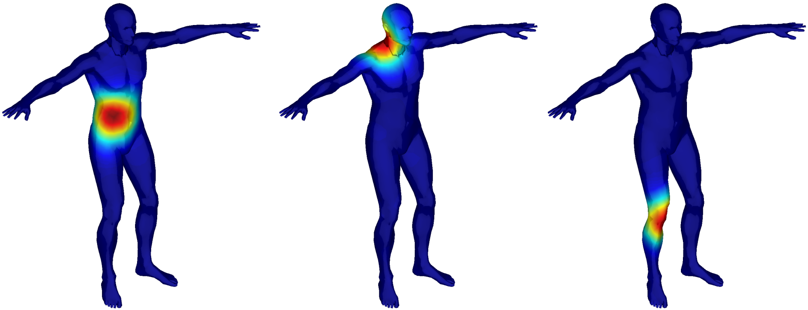
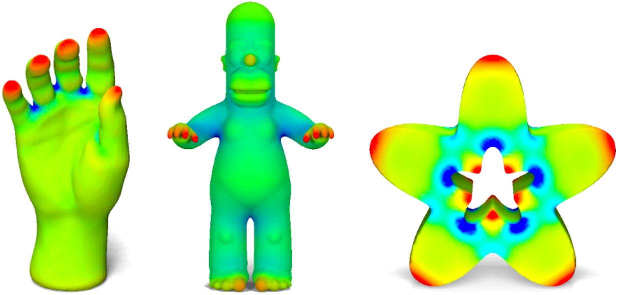
- "impulse response" to a delta-function at ξ
- "how much heat is transferred from point x to ξ in time t"

Heat kernel



Heat kernels at different points
(rows/columns of matrix $e^{-t\Delta_X}$)

Autodiffusivity



Autodiffusivity = diagonal of matrix $e^{-t\Delta_X}$

Related to Gaussian curvature by virtue of the Taylor expansion

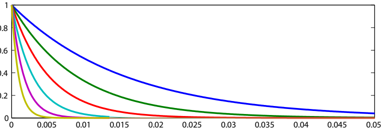
$$h_t(x,x) \approx \frac{1}{4\pi t} + \frac{K(x)}{12\pi} + \mathcal{O}(t)$$

Sun, Ovsjanikov, Guibas 2009

Spectral descriptors

$$\mathbf{f}(x) = \sum_{k \geq 1} \begin{pmatrix} \tau_1(\lambda_k) \\ \vdots \\ \tau_Q(\lambda_k) \end{pmatrix} \phi_k^2(x)$$

Heat Kernel Signature (HKS)

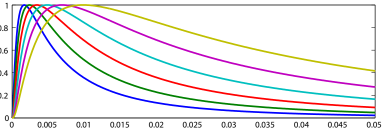


Low-pass filter bank

$$\tau_i(\lambda) = \exp(-\lambda t_i)$$

Heat autodiffusivity

Wave Kernel Signature (WKS)



Band-pass filter bank

$$\tau_i(\lambda) = \exp\left(-\frac{(\log e_i - \log \lambda)^2}{\sigma^2}\right)$$

Probability of a quantum particle

Sun, Ovsjanikov, Guibas 2009; Aubry, Schlickewei, Cremers 2011