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Symmetry in Shapes – Theory and Practice

Intrinsic Symmetry Detection

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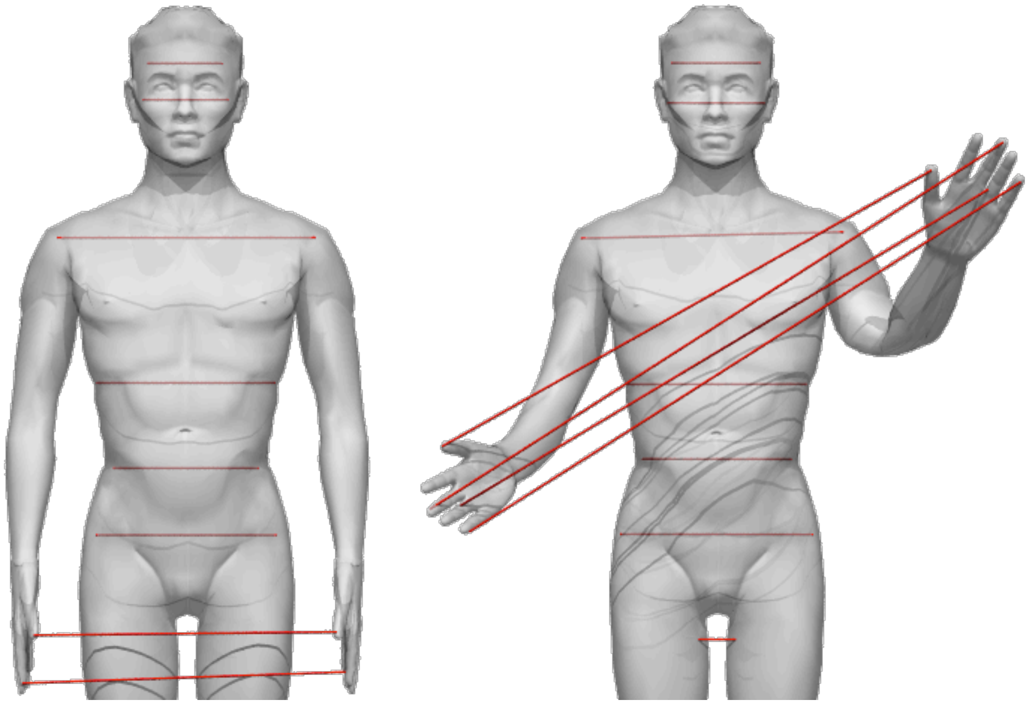
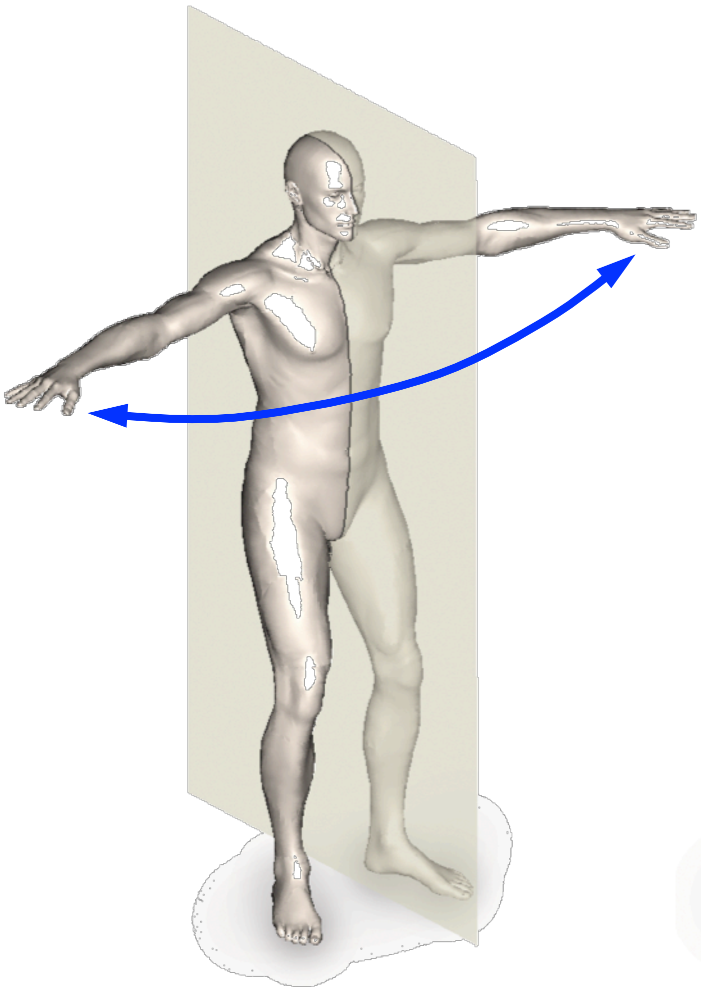


Intrinsic Symmetries

Intuition



I am symmetric.

Bronstein et al.

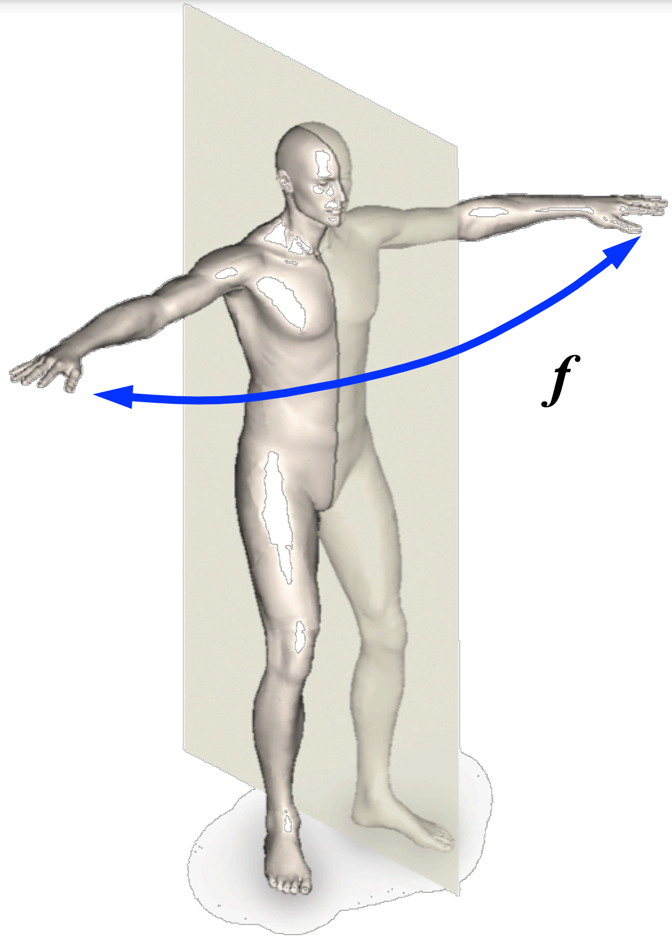
What about us?

Problem Formulation

- Shape X is **symmetric**, if there exists a **transformation** f such that $f(X) = X$.

What class of transformations is allowed?

- **Extrinsic**:
  f is a combination of:
  - Rotation,
  - Translation,
  - Reflection,
  - (Scaling)



Bronstein et al.

Problem Formulation

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  a **transformation** f
  such that $f(X) = X$.

What class of transformations is allowed?

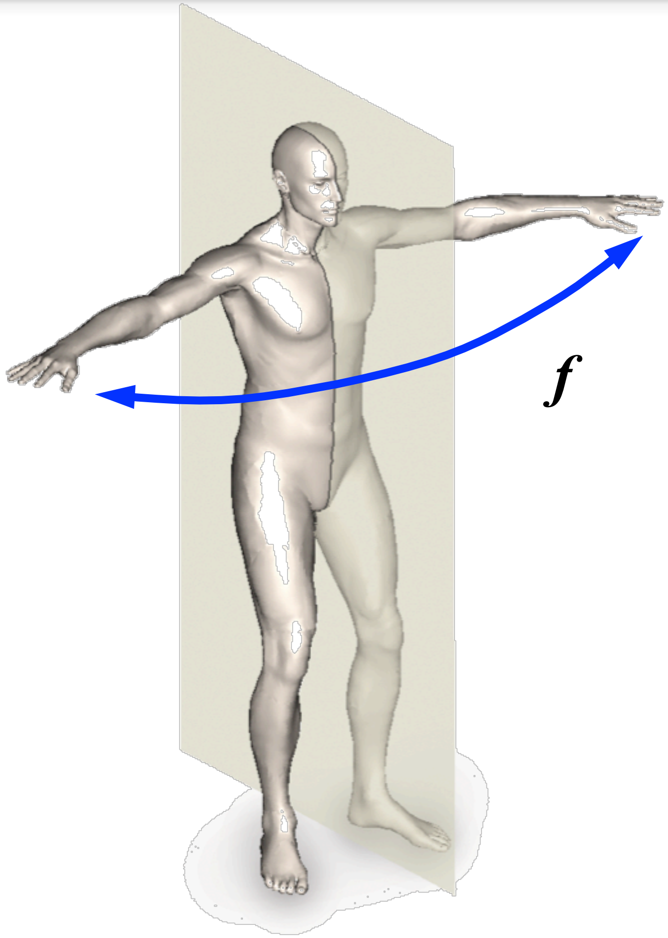
- **Extrinsic**:
  f is: rotation, translation, reflection
- **Intrinsic?**

Bronstein et al.

Problem Formulation

Fundamental Theorem:

A map $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a combination of translation, rotation, and reflection if and only if it preserves all Euclidean distances.



Problem Formulation

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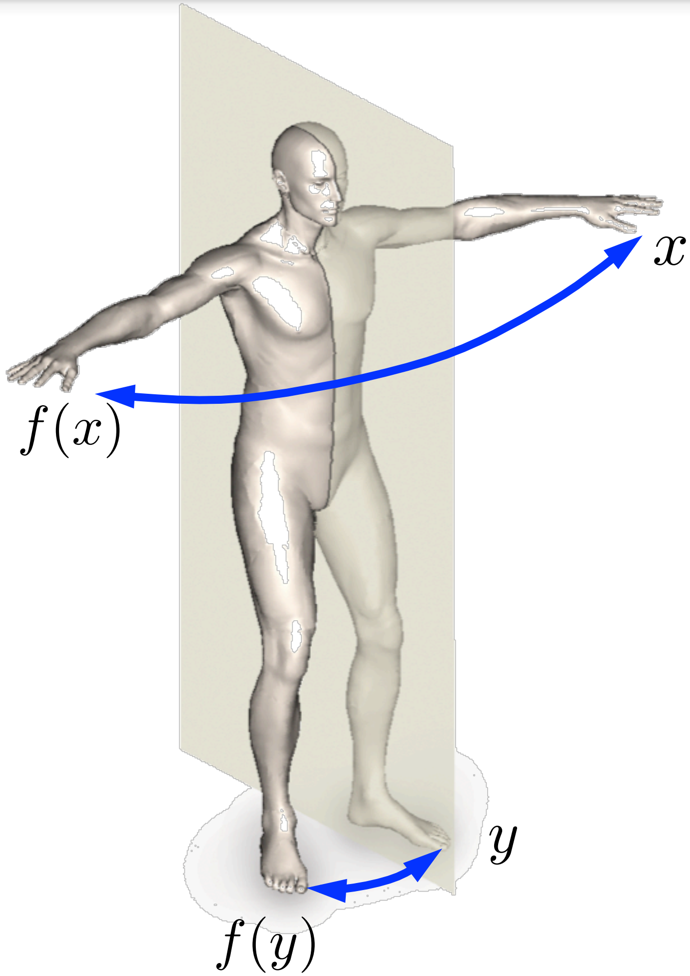
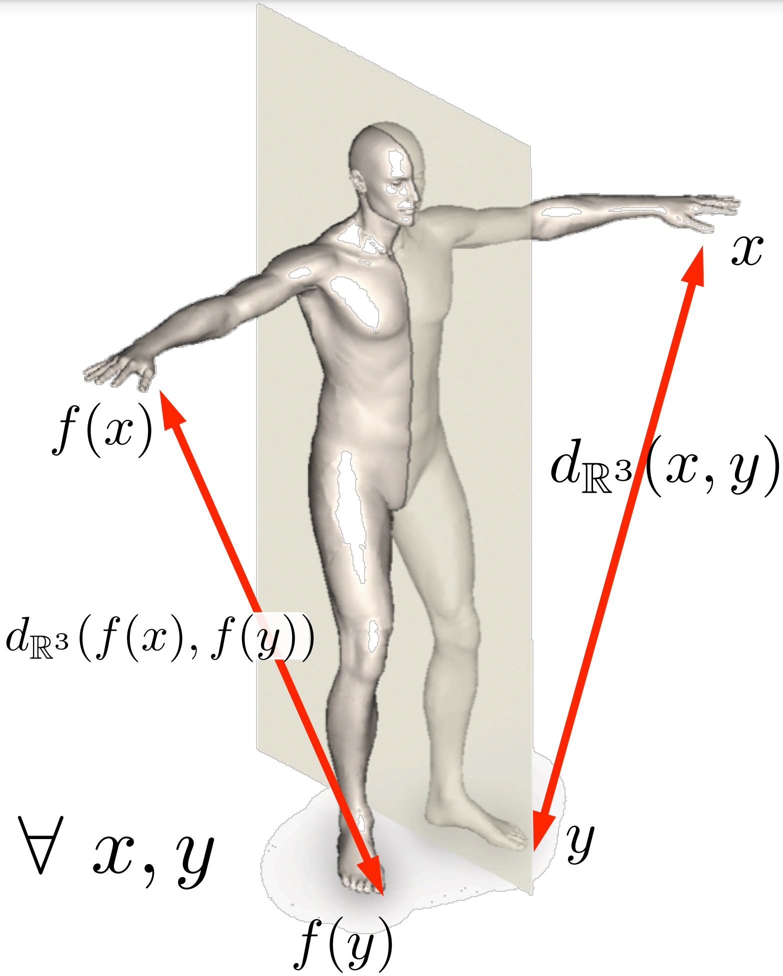


Problem Formulation

Fundamental Theorem:

A map $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a combination of translation, rotation, and reflection if and only if it preserves all Euclidean distances.

$$d_{\mathbb{R}^3}(f(x), f(y)) = d_{\mathbb{R}^3}(x, y) \ \forall \ x, y$$



Extrinsic Formulation

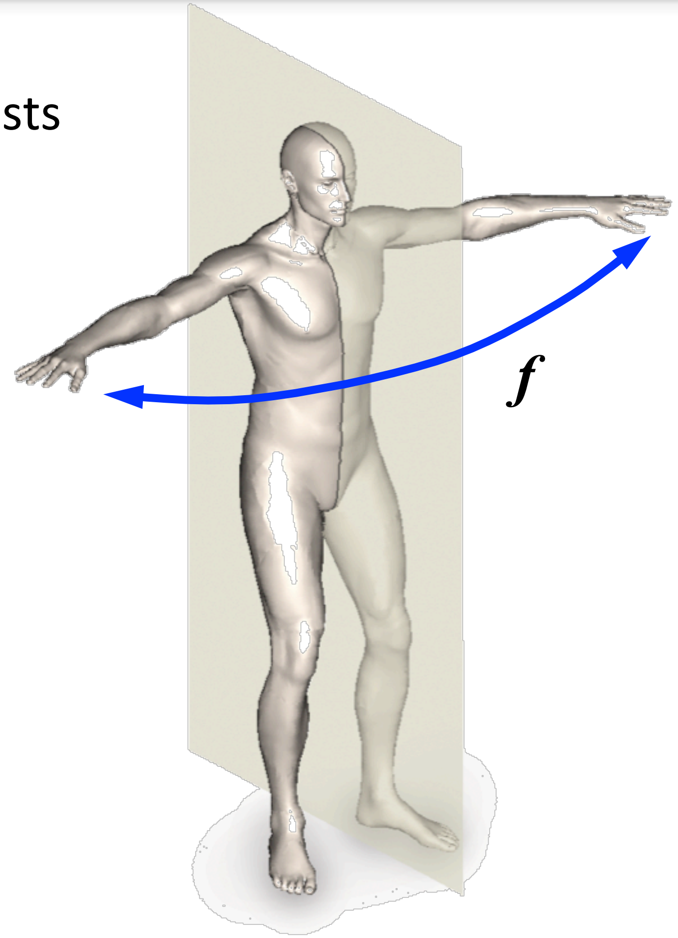
- Shape X is **extrinsically symmetric**, if there exists
  a **rigid motion** f,
  such that $f(X) = X$.

Equivalently:

- Shape X is **extrinsically symmetric**,
  if there exists a map:

$$f : X \to X \text{ s.t.}$$

$$d_{\mathbb{R}^3}(f(x), f(y)) = d_{\mathbb{R}^3}(x, y) \; \forall \; x, y$$



Bronstein et al.

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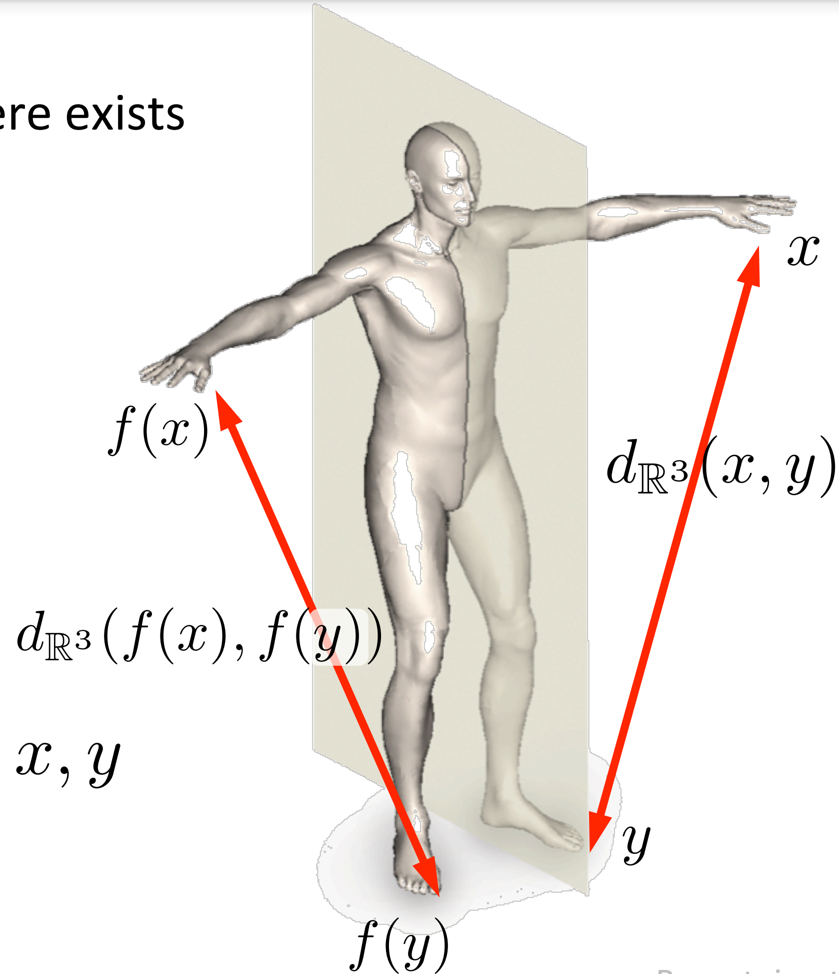


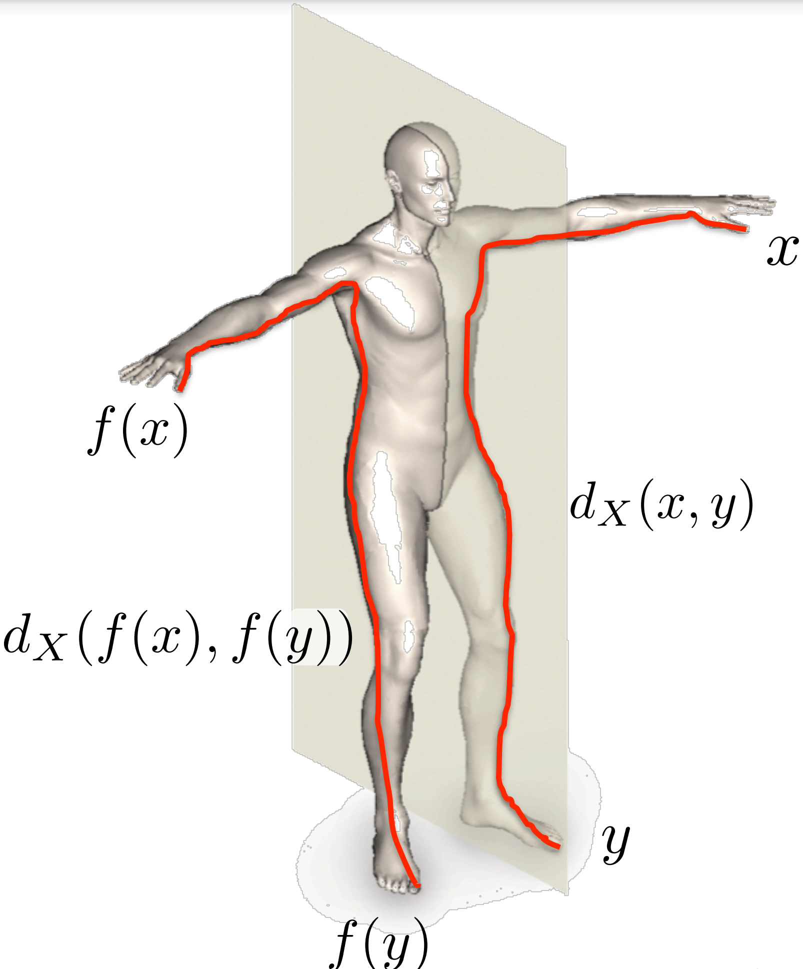
Bronstein et al.

Intrinsic Formulation

- Shape X is **intrinsically symmetric**, if there exists a map:

$$f : X \to X \text{ s.t.}$$

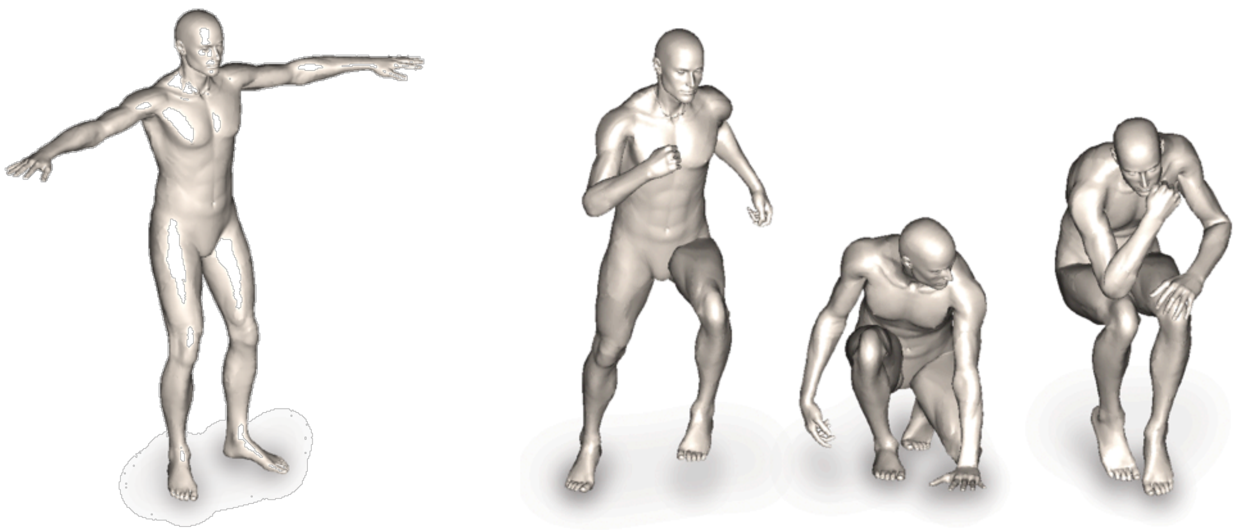
$$d_{X}(f(x), f(y)) = d_{X}(x, y) \; \forall \; x, y$$



Bronstein et al.

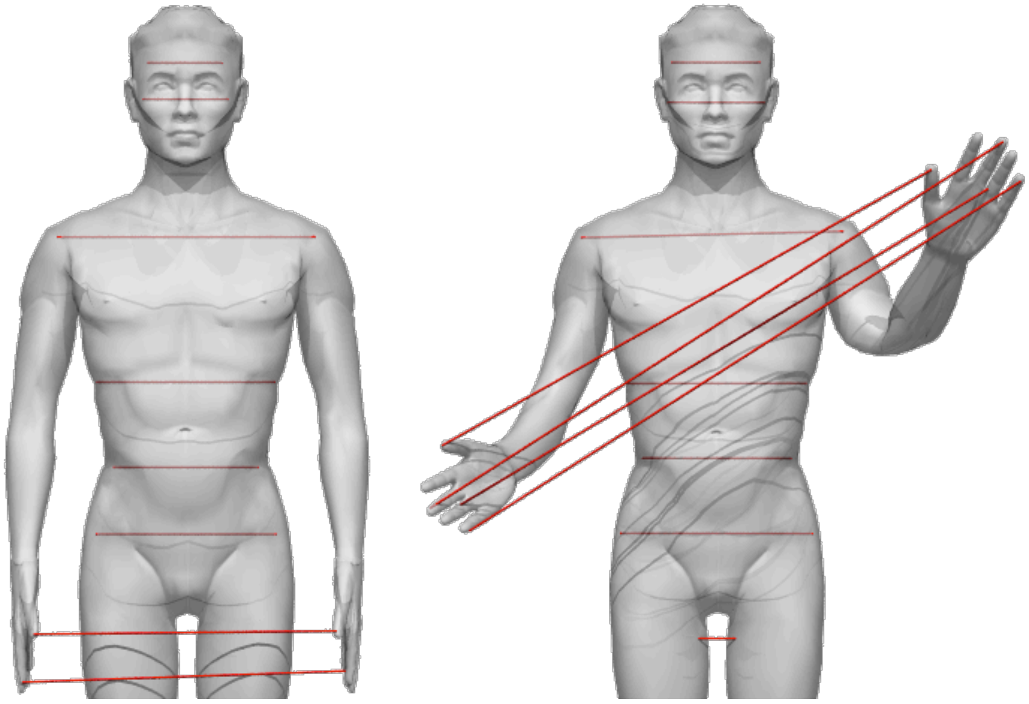
Intrinsic Formulation

- **Intrinsic Isometries**:
  Shape deformations that preserve intrinsic (geodesic) distances.



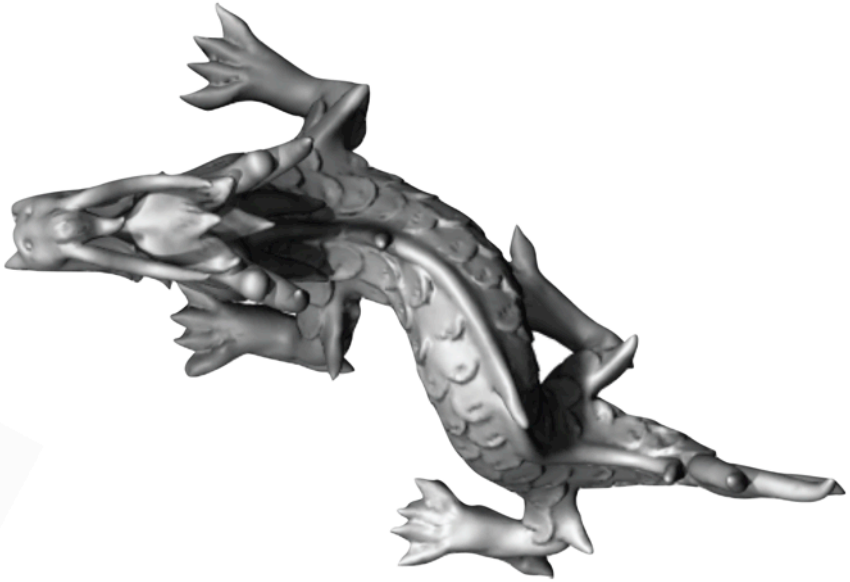
Intrinsic Formulation

- **Intrinsic Symmetries**:
  Self-maps that approximately preserve geodesic distances



Intrinsic Formulation

Extrinsic symmetries depend on the embedding of the object in space.



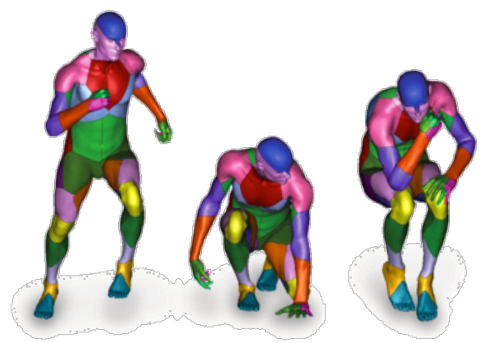
Intrinsic symmetries are defined with respect to an intrinsic metric of the surface.

Intrinsic Symmetry Detection

Idea:

1. Solve the optimization problem directly:

$$\min_{f:X\to X} \sum_{x,x'\in X} \left(d_X(x,x') - d_X(f(x),f(x'))\right)^2$$



Approach:

GMDS: treat each point as a variable, solve using nonlinear optimization (main difficulty: obtaining the gradient of the energy).

Raviv et al., **Symmetries of Non-Rigid Shapes**., *NRTL* 2007, IJCV 2009

Intrinsic Symmetry Detection

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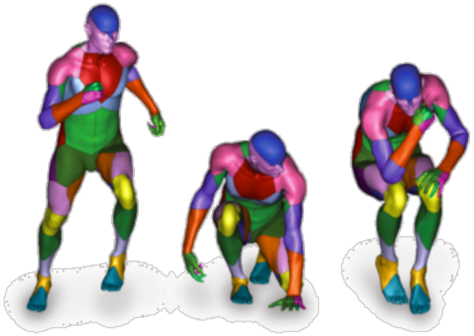


Difficulties:

1. Energy is non-linear non-convex, need a good initial guess.
2. Optimization is expensive (compute over a small number of points).
3. Want to stay away from the trivial solution.

Raviv et al., **Symmetries of Non-Rigid Shapes**., *NRTL* 2007, IJCV 2009

Intrinsic Symmetry Detection

Initial Guess:

1. Adapt the Global Rigid Matching idea to non-rigid setting:
    1. For each point on the surface find a **non-rigid descriptor**.
    2. Match points with similar descriptors.
    3. Merge disjoint pairs of correspondences into sets of 4.
    4. Compute the distortion of the partial solution.

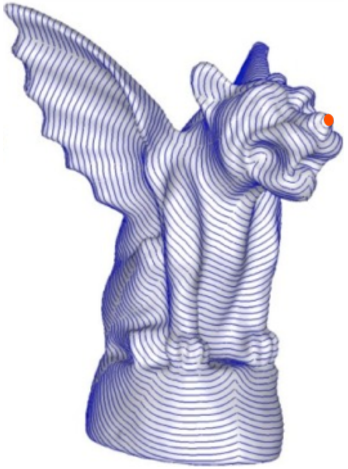
2. Branch and bound global optimum
    1. Incrementally add points to get a partial solution.
    2. If the distortion is greater than the known solution, disregard it.
    3. Depends on the quality of the initial greedy guess.

Raviv et al., **Symmetries of Non-Rigid Shapes**., *NRTL* 2007, IJCV 2009

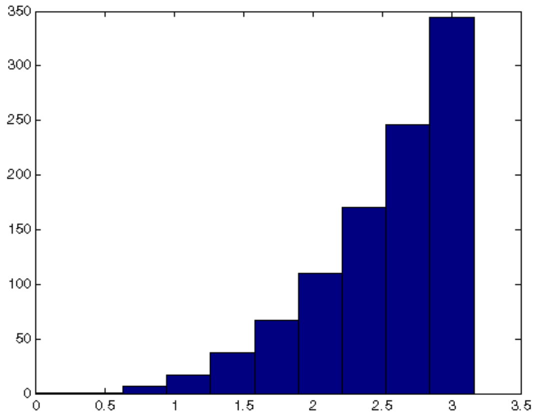
Intrinsic Symmetry Detection

Non-rigid Descriptor:

1. At each point compute the histogram of geodesic distances.



Geodesic level sets



How many points within each level set

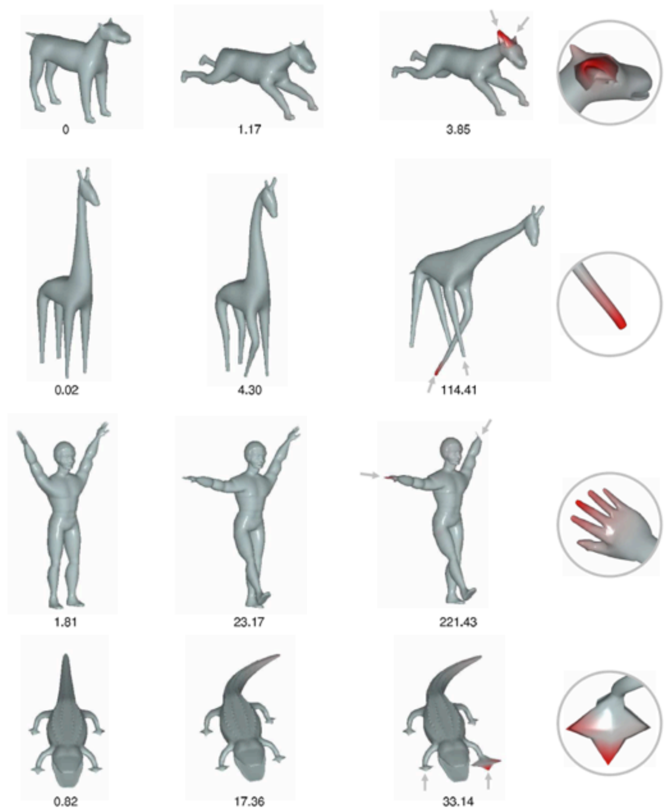
Comparing Descriptors:

1. Non-trivial. Comparing $\|h_i - h_j\|_2$ bad because of binning. Use instead:

$$d(h_i, h_j) = \sqrt{(h_i - h_j)^{\mathrm{T}} A (h_i - h_j)}$$ where A_{mn} distance between bins.

Intrinsic Symmetry Detection

Results:



Limitations:

1. Not easy to explore *multiple* symmetries.

2. Need a better descriptor.

Intrinsic Symmetry Detection

- Purely algebraic method for detecting intrinsic symmetries, and point-to-point correspondences.
- Grouping symmetries into discrete classes.
- Main Observation: In a certain space, intrinsic symmetries become extrinsic symmetries.

O., Sun, Guibas, **Global Intrinsic Symmetries of Shapes**, SGP 2008

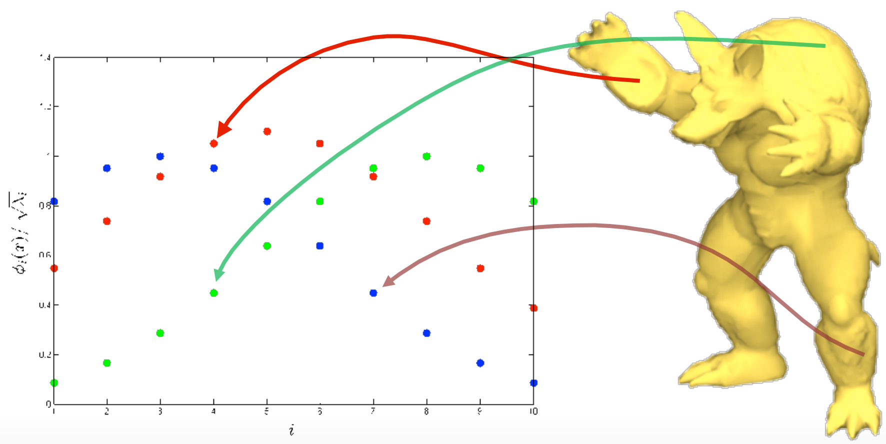
Global Point Signatures

- Given a point x on the surface, its GPS signature:

$$s(x) = \left(\frac{\phi_1(x)}{\sqrt{\lambda_1}}, \frac{\phi_2(x)}{\sqrt{\lambda_2}}, \ldots, \frac{\phi_i(x)}{\sqrt{\lambda_i}}, \ldots \right)$$

Rustamov, 2007

Where $\phi_i(x)$ is the value of the eigenfunction of the Laplace-Beltrami operator at x.

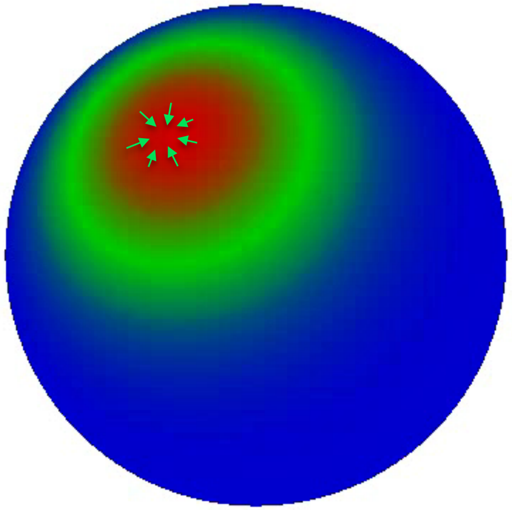


Laplace-Beltrami Operator

Given a compact Riemannian manifold X without boundary, the Laplace-Beltrami operator:

$$\Delta : C^\infty(X) \to C^\infty(X), \Delta f = \mathrm{div}\nabla f$$

$$\frac{\partial f}{\partial t}\ \mathrm{div}\Delta f$$



Laplace-Beltrami Operator

Given a compact Riemannian manifold X without boundary, the Laplace-Beltrami operator $\triangle$:

1. Is invariant under isometric deformations.

2. Characterizes the manifold completely.

3. Has a countable eigendecomposition:

$$\Delta\phi_i = \lambda_i\phi_i$$

   that forms an orthonormal basis for $L^2(X)$.

Observations

GPS(X)

$$s(x) = \left(\frac{\phi_1(x)}{\sqrt{\lambda_1}}, \frac{\phi_2(x)}{\sqrt{\lambda_2}}, \ldots, \frac{\phi_i(x)}{\sqrt{\lambda_i}}, \ldots\right)$$

Theorem:

If X has an intrinsic symmetry $f : X \to X$, then GPS(X) has a Euclidean symmetry. I.e.:

$$\|s(x) - s(x')\|_2 = \|s\left(f(x)\right) - s\left(f(x')\right)\|_2 \;\; \forall \;\; x, x' \in X$$

Moreover, restriction to each distinct eigenvalue is symmetric.

O., Sun, Guibas, **Global Intrinsic Symmetries of Shapes**, SGP 2008

Observations

Theorem:

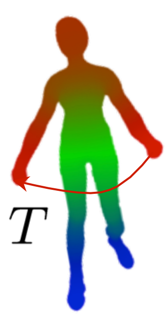
If M has an intrinsic symmetry $T : M \to M$, then GPS(M) has a Euclidean symmetry. I.e.:

$$\|s(x) - s(y)\|_2 = \|s(T(x)) - s(T(y))\|_2 \; \forall \; x, y \in M$$

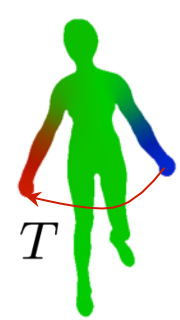
Moreover, restriction to each distinct eigenvalue is symmetric.

For non-repeating eigenvalues, only 2 possibilities:

Positive



or



Negative

$$\phi_i(T(x)) = \phi_i(x) \; \forall \; x \in M$$

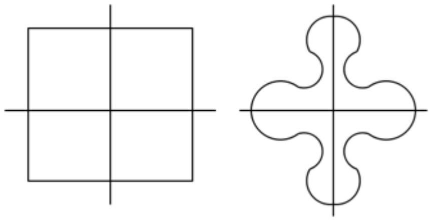
$$\phi_i(T(x)) = -\phi_i(x) \; \forall \; x \in M$$

Restricted Signature Space

- Only include non-repeating eigenvalues.

- In the restricted space, intrinsic symmetries are reflective symmetries around principal axes:

$$|s_i(f(x))| = |s_i(x)|$$



- Detecting such symmetries is easy. Find which coordinates to flip. For each point x, is there a correspondence y, s.t.

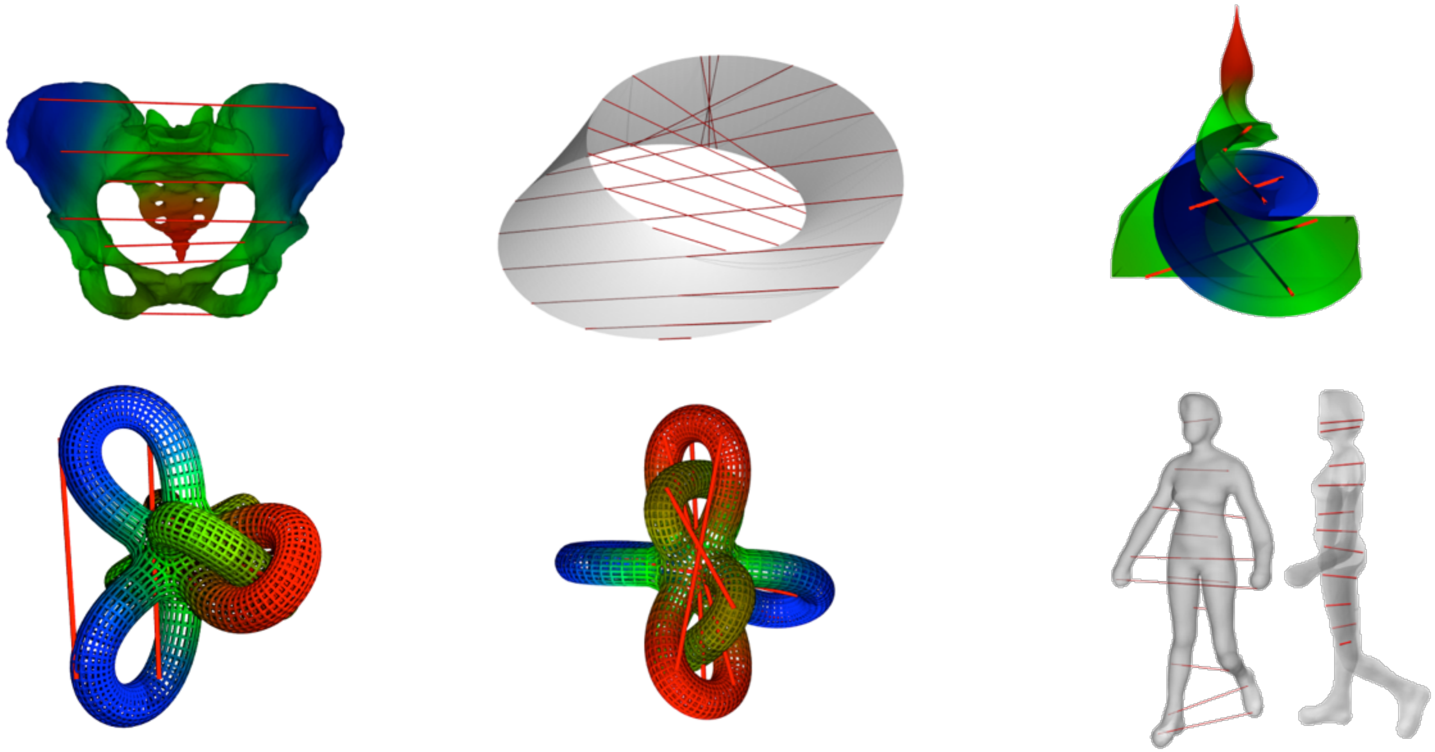
$$|s_j(y)| = |s_j(x)| \;\; \forall j \neq i, \text{ and } s_i(y) = -s_i(x) \quad ?$$

- Only need nearest neighbor computation in high d.

Implementation

- Use Belkin et al. *Discrete Laplace operator on meshed surfaces*, SOCG 2008 for eigenfunction computations.
- ANN library for nearest neighbor computations.
- Query in KD-tree depends on the dimension of the data. GPS is homeomorphism, so dimension = 2. No curse of dimensionality with increasing *d.*
- Overall complexity $O(d^3 n \log n)$.

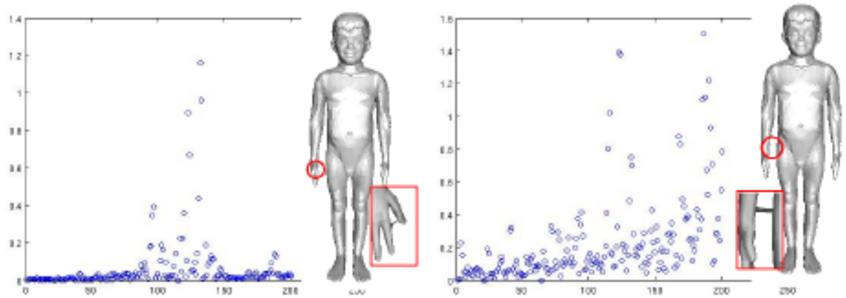
Results



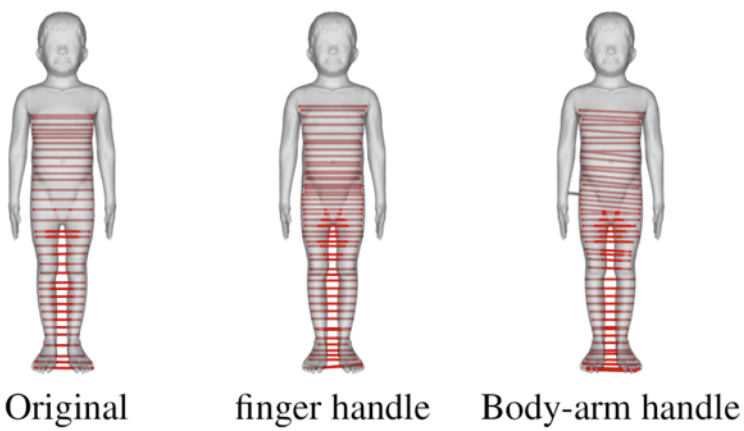
- Euclidean symmetries when present.
- Two different symmetries for human shape.

Topological Noise

- Change in GPS after geodesic shortcuts:

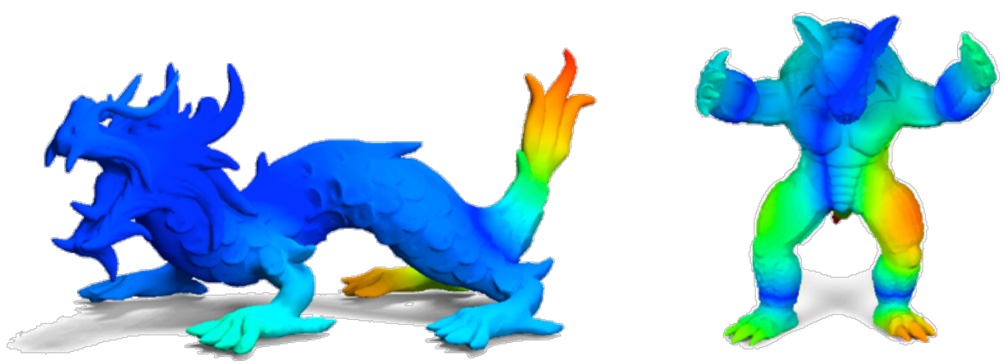


- Correspondences



Limitations

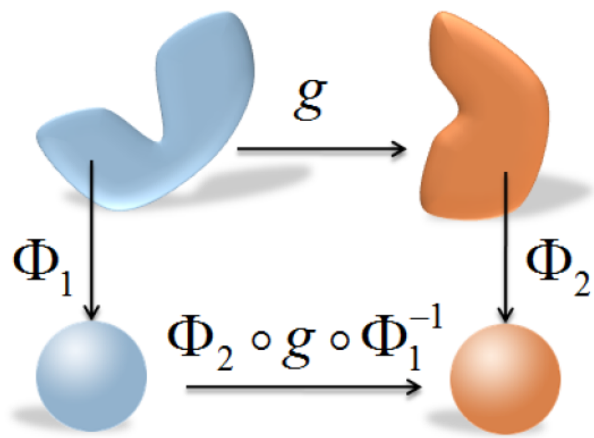
- Can only detect very global symmetries.
- Cannot handle *continuous* symmetries.
- In the discrete setting even non-repeating eigenfunctions can be unstable

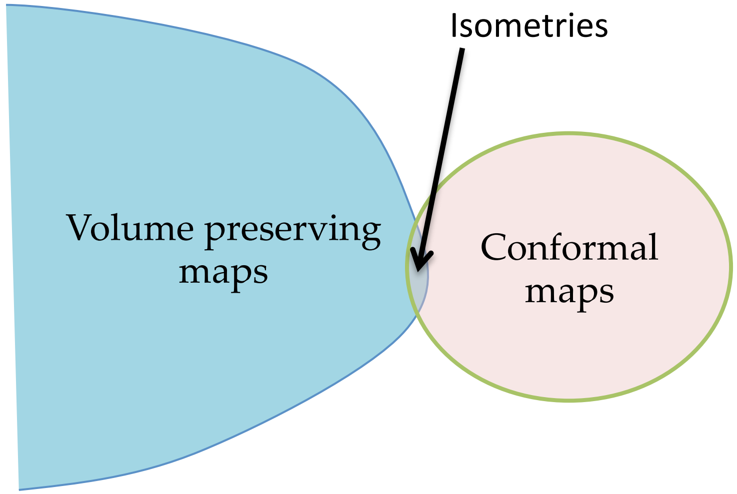


O., Sun, Guibas, **Global Intrinsic Symmetries of Shapes**, SGP 2008

Intrinsic Symmetry Detection

- Möbius Voting:



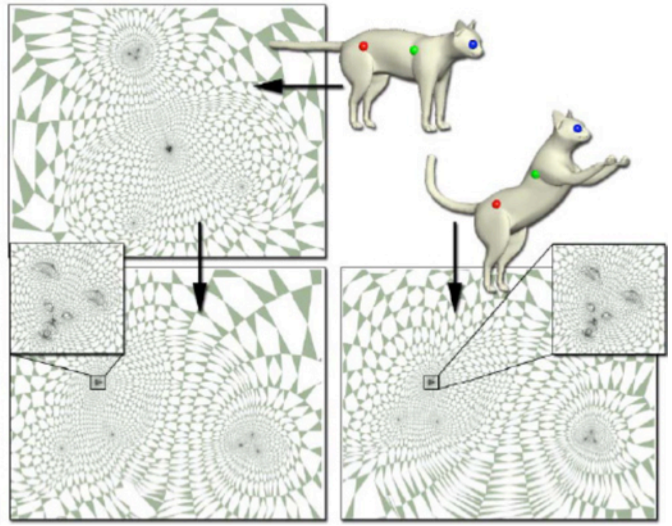


Lipman and Funkhouser SIGGRAPH'09

Isometries are a subgroup of the group of conformal maps. For genus zero surfaces: 3 correspondences constrain all degrees of freedom, and the optimal transformation has a closed form solution.

Intrinsic Symmetry Detection

- Möbius Voting for shape matching:



Isometries are a subgroup of the group of conformal maps. For genus zero surfaces: 3 correspondences constrain all degrees of freedom, and the optimal transformation has a closed form solution.

Intrinsic Symmetry Detection

- Möbius Voting-based symmetry detection:

1) Map the mesh surface to the extended complex plane $\hat{\mathbb{C}}$.
2) Generate a set of anti-Mobius transformations.
3) Measure alignment score
4) Return the best alignment

Iterate (steps 2–3)

Kim, Lipman, Chen, and Funkhouser **Mobius Transformations for Global Intrinsic Symmetry Analysis**, SGP 2010

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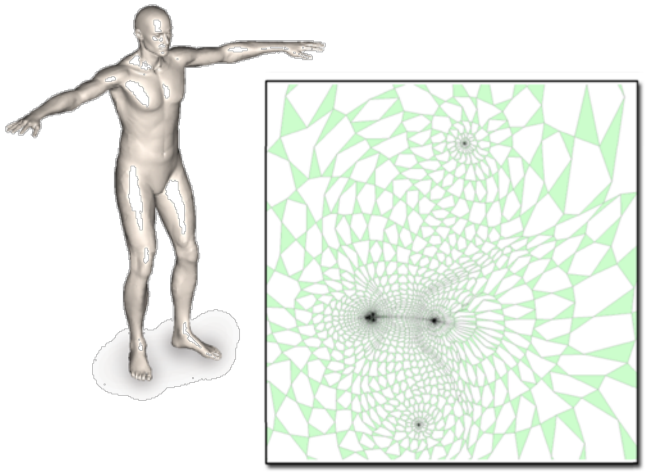

Mid-point uniformisation
(Lipman et al. '09)

Conformal mapping onto the sphere by solving a sparse linear (Laplacian) system

Kim, Lipman, Chen, and Funkhouser **Mobius Transformations for Global Intrinsic Symmetry Analysis**, SGP 2010

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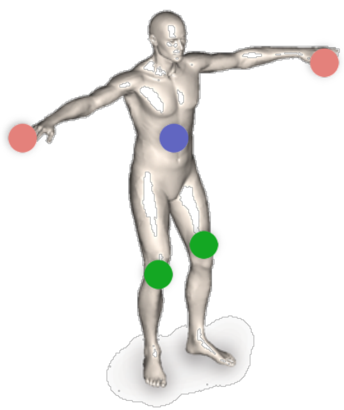

Find likely triplets of correspondences

Use intrinsic *symmetry-invariant descriptors.*

Kim, Lipman, Chen, and Funkhouser **Mobius Transformations for Global Intrinsic Symmetry Analysis**, SGP 2010

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Use the initial triplet to find correspondences between all other points.

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Closed form solution in the extended complex plane embedding.

Kim, Lipman, Chen, and Funkhouser **Mobius Transformations for Global Intrinsic Symmetry Analysis**, SGP 2010

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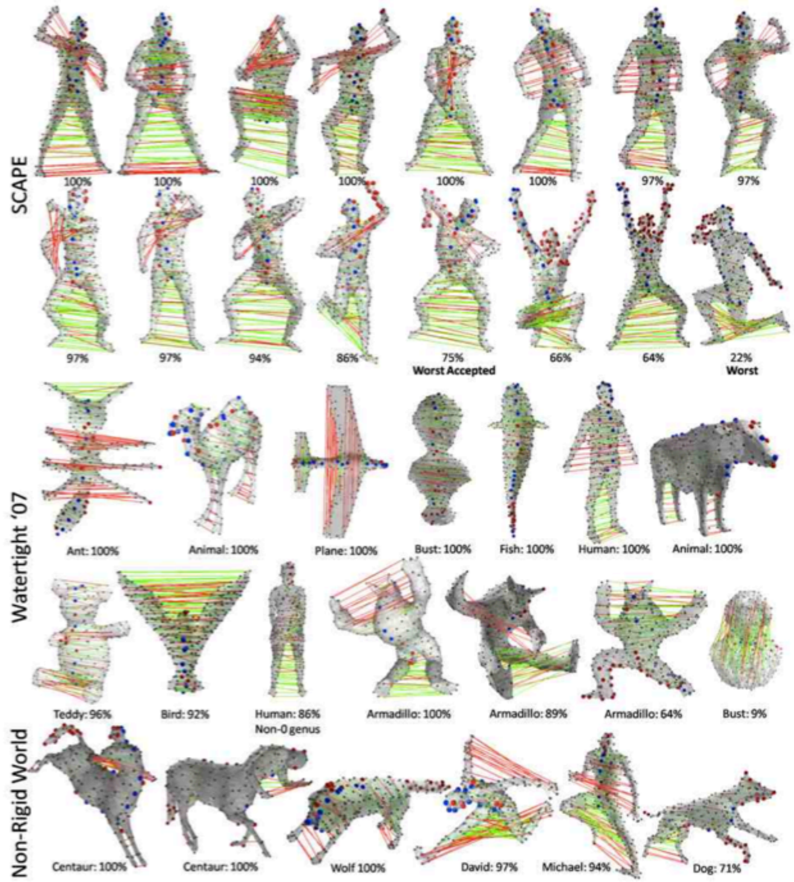
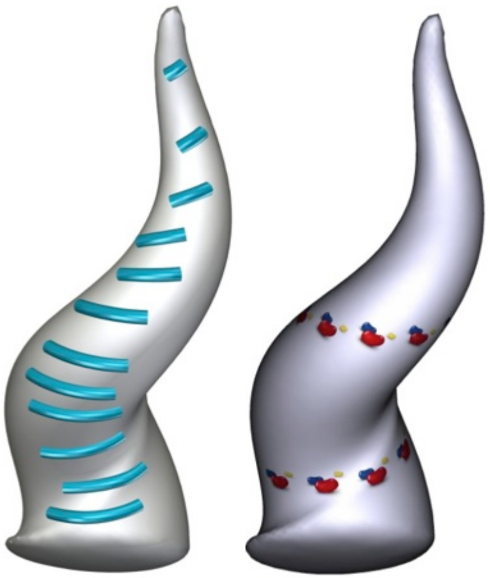
Results



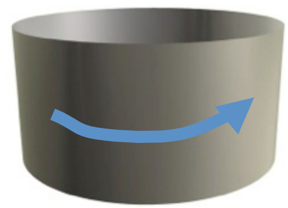
- Largest-scale evaluation of an intrinsic symmetry-detection method.
- Benchmark for comparing other methods.

Continuous Intrinsic Symmetries



Ben-Chen, Butscher, Solomon, Guibas **On discrete killing vector fields and patterns on surfaces**, SGP 2010

Continuous Intrinsic Symmetries

	Extrinsic	Intrinsic Approximate
Continuous		
Discrete		
Representation	Rigid Transformations	?

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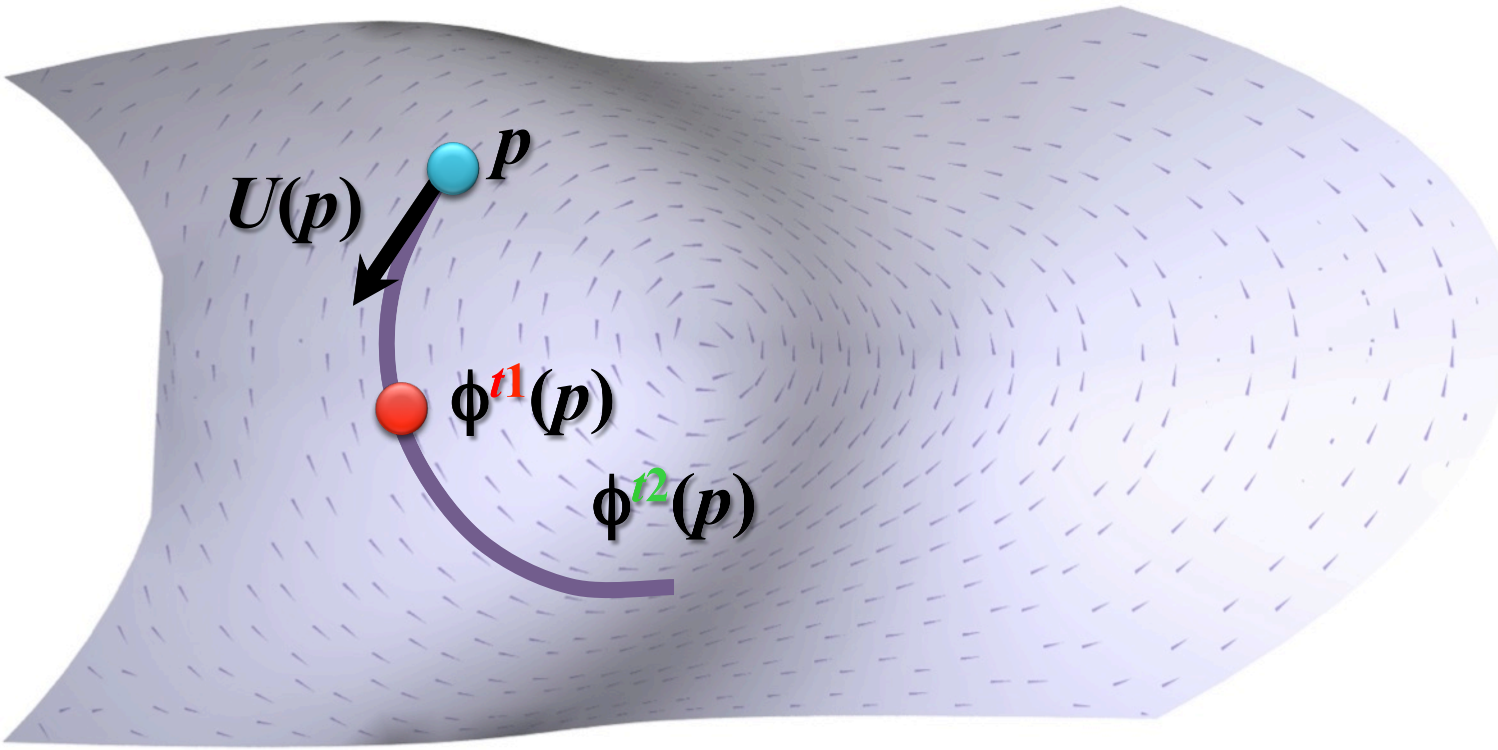
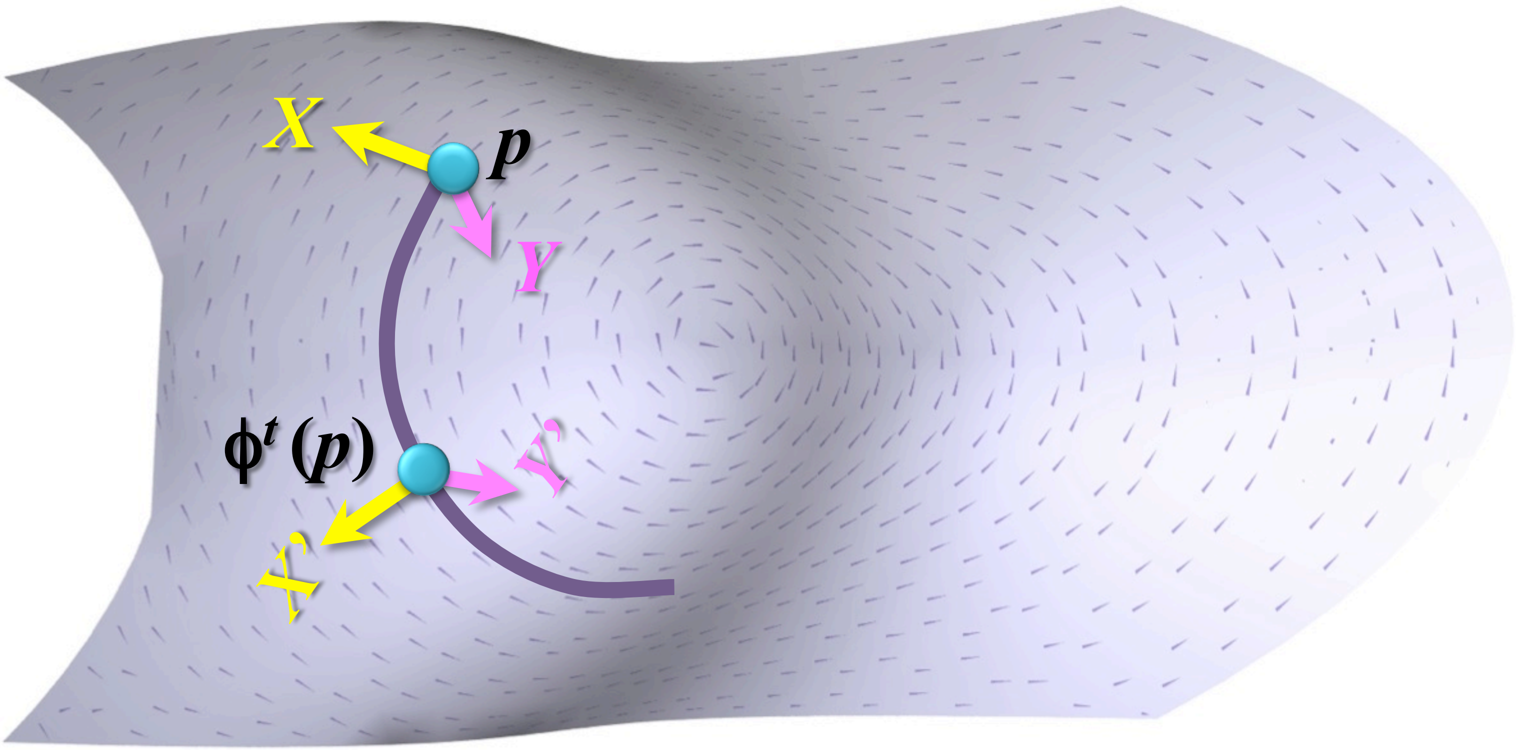
Represent Transformations using Tangent Vector Fields



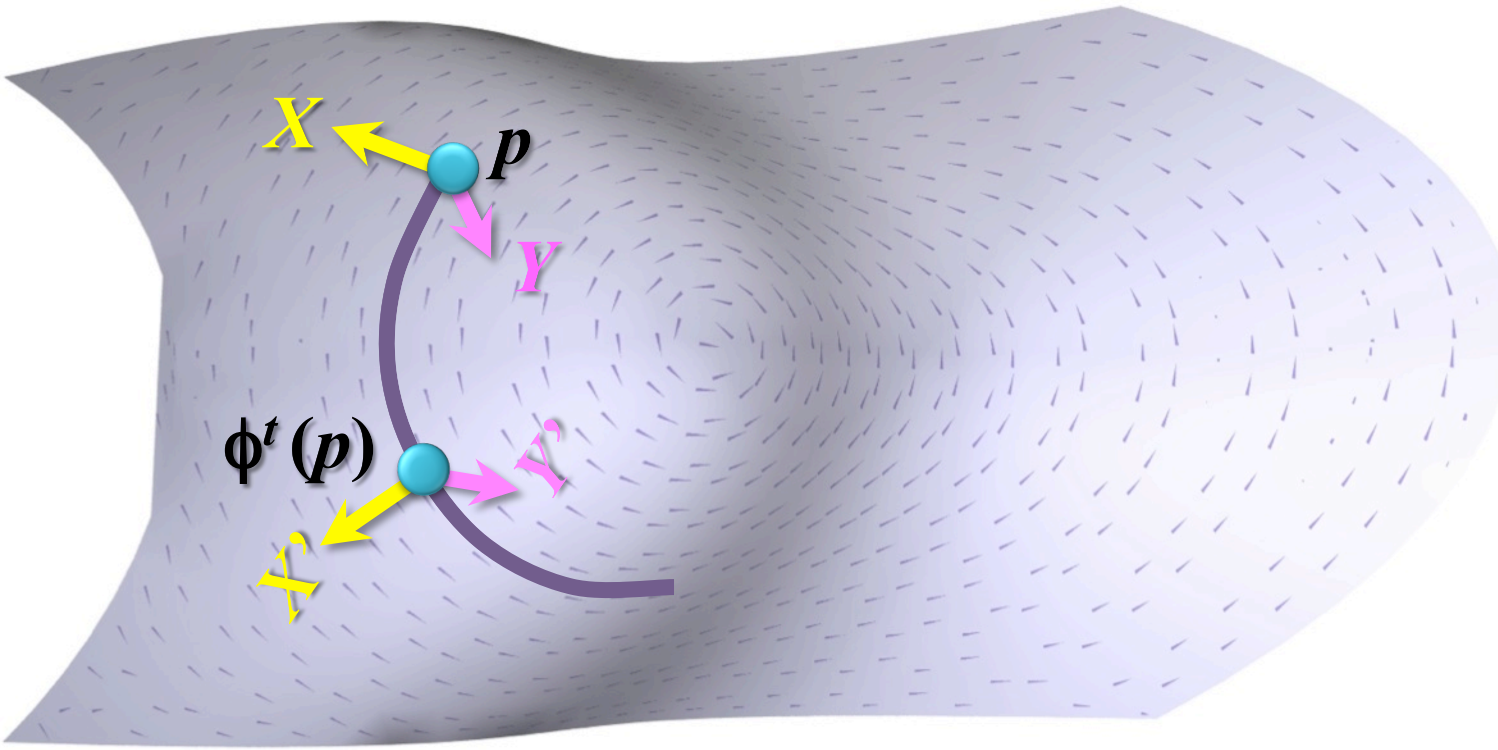
$\phi^t(p)$ – One-parameter family of mappings generated by the tangent vector field U

Preserve the Metric



$$g_p(X,Y)$$

Preserve the Metric



$$\lim_{t \to 0} \frac{g_{\phi^t(p)}(X',Y') - g_p(X,Y)}{t} = 0$$

Killing Vector Fields

Vector fields whose flow preserves the metric

U is a KVF if for any X, Y:

$$\mathcal{L}_U g = \lim_{t \to 0} \frac{g_p(X,Y) - g_{\phi^t(p)}(X',Y')}{t} = 0$$

Killing Vector Fields (again)

The Killing Equation

- U is a KVF is and only if for every V:

$$g_p\left(\nabla_V U, V\right) = 0$$

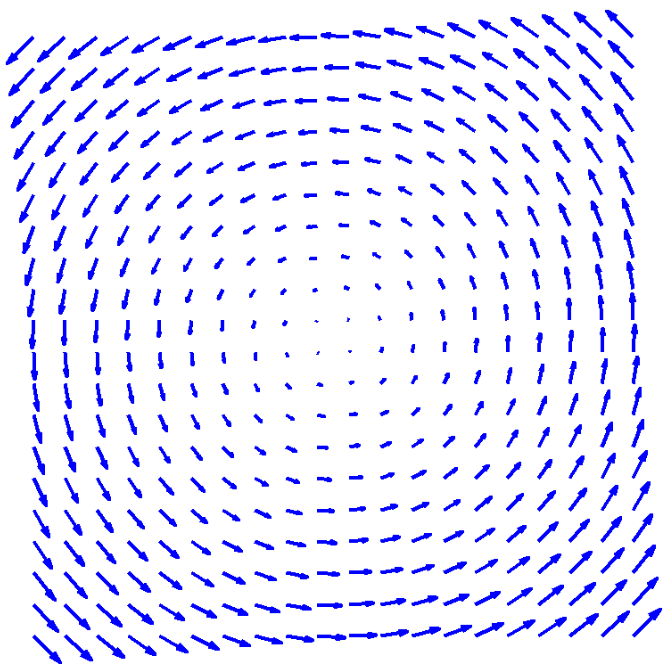
Covariant Derivative — Killing Vector Field — Any Vector

- In R^n means: $\left\langle \nabla U \cdot V, V \right\rangle = 0$

Killing Vector Fields

A (very) simple example

$$U = (u(x,y), v(x,y)) = (-y, x)$$



$$\nabla U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$V = \left(v_x, v_y\right)$$

$$\nabla U \cdot V = \left(-v_y, v_x\right)$$

Back to the Killing Equation

$$\forall V \quad \langle \nabla U \cdot V, V \rangle = 0$$

Equivalent to:

$$\nabla U + \nabla U^T = 0$$

R^n : ∇U = Jacobian matrix

Surface: ∇U = covariant derivative tensor

Computing AKVFs

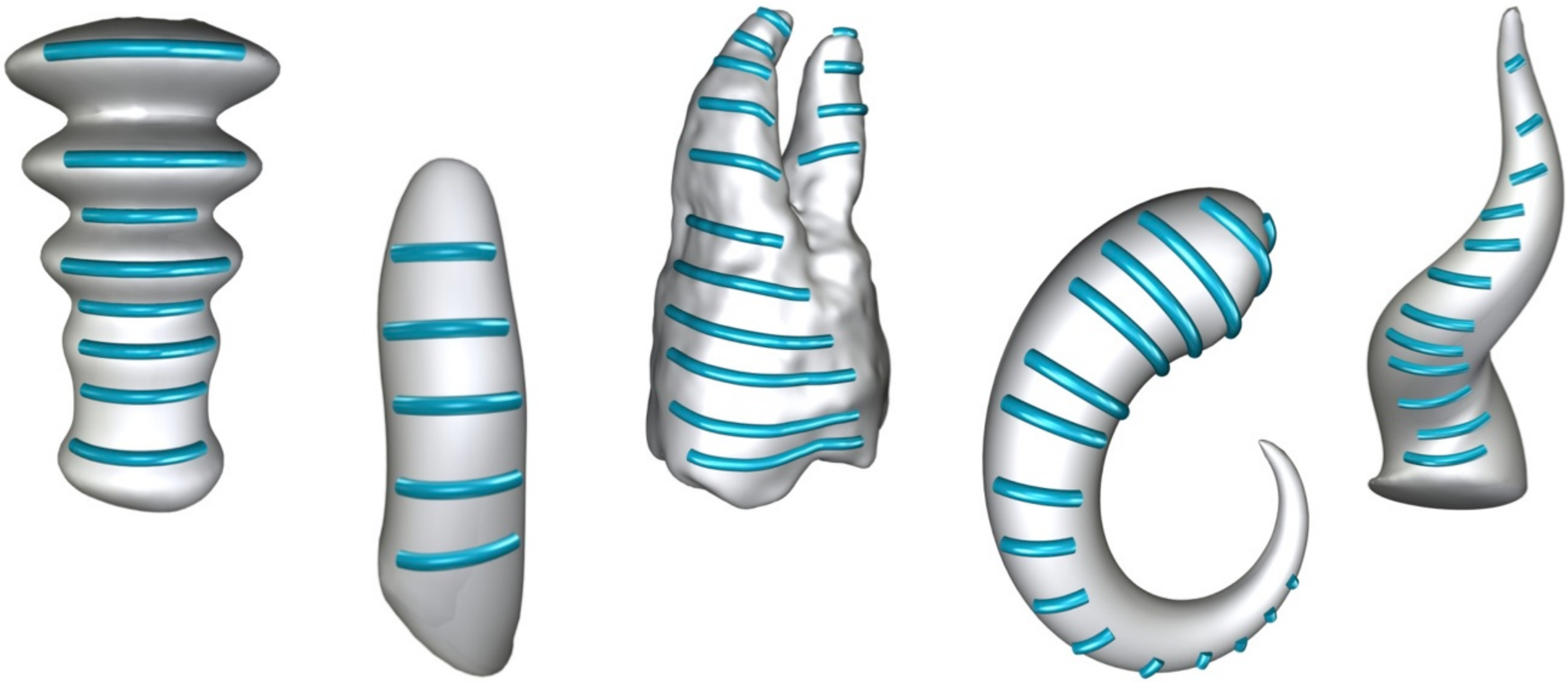
Solve:

$$\min_U E_K(U) = \int_M \left|\nabla U + \nabla U^T\right|^2 dv \quad s.t. \int_M |U|^2 dv = 1$$

On a triangulated mesh.

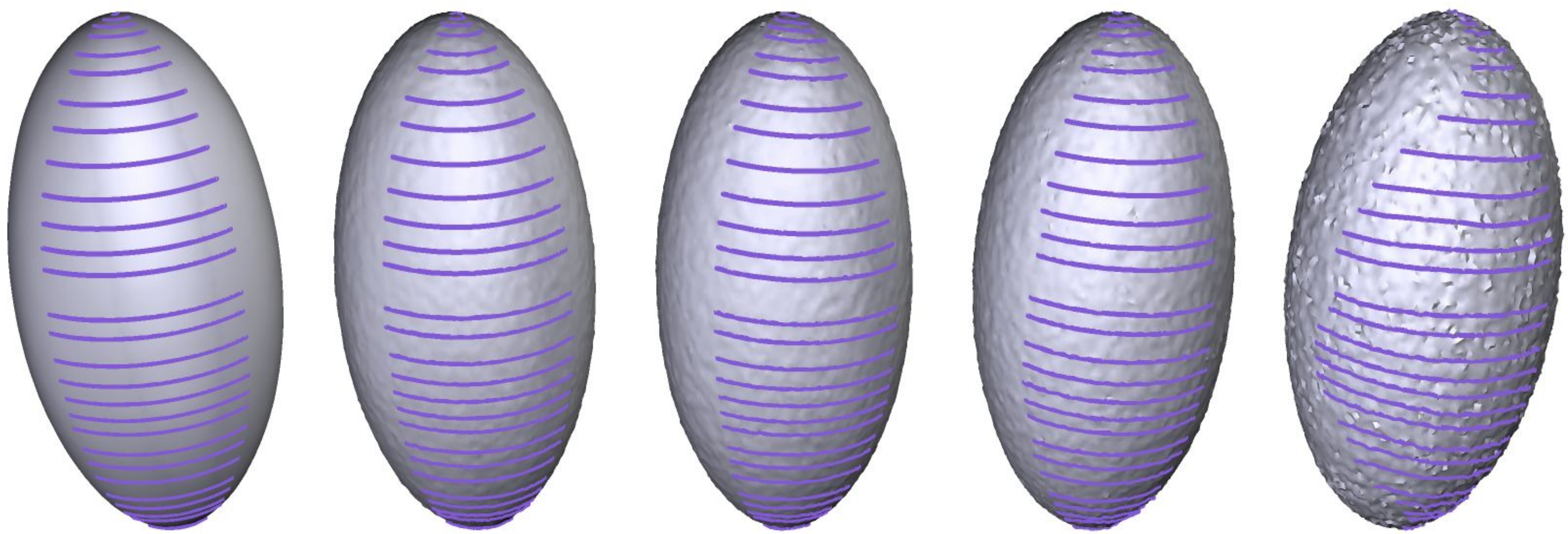
Reformulate using (Discrete) Exterior Calculus. Leads to an eigendecomposition problem.

AKVFs in the Wild



Ben-Chen, Butscher, Solomon, Guibas **On discrete killing vector fields and patterns on surfaces**, SGP 2010

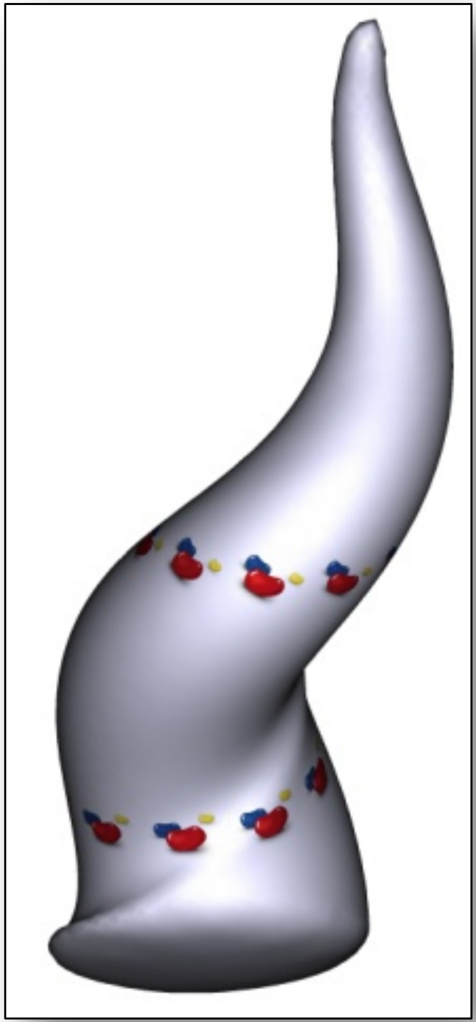
Approximate KVFs
Noise

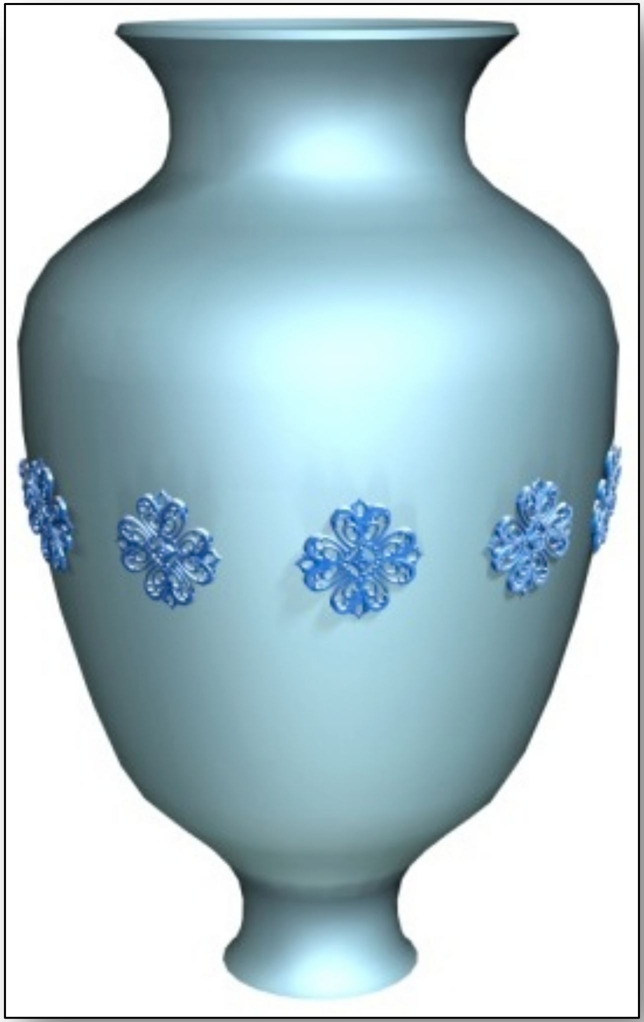


$\sigma = 0.065$ $E = 0.29$ | $\sigma = 0.087$ $E = 0.55$ | $\sigma = 0.1145$ $E = 1.33$ | $\sigma = 0.2$ $E = 6.7$

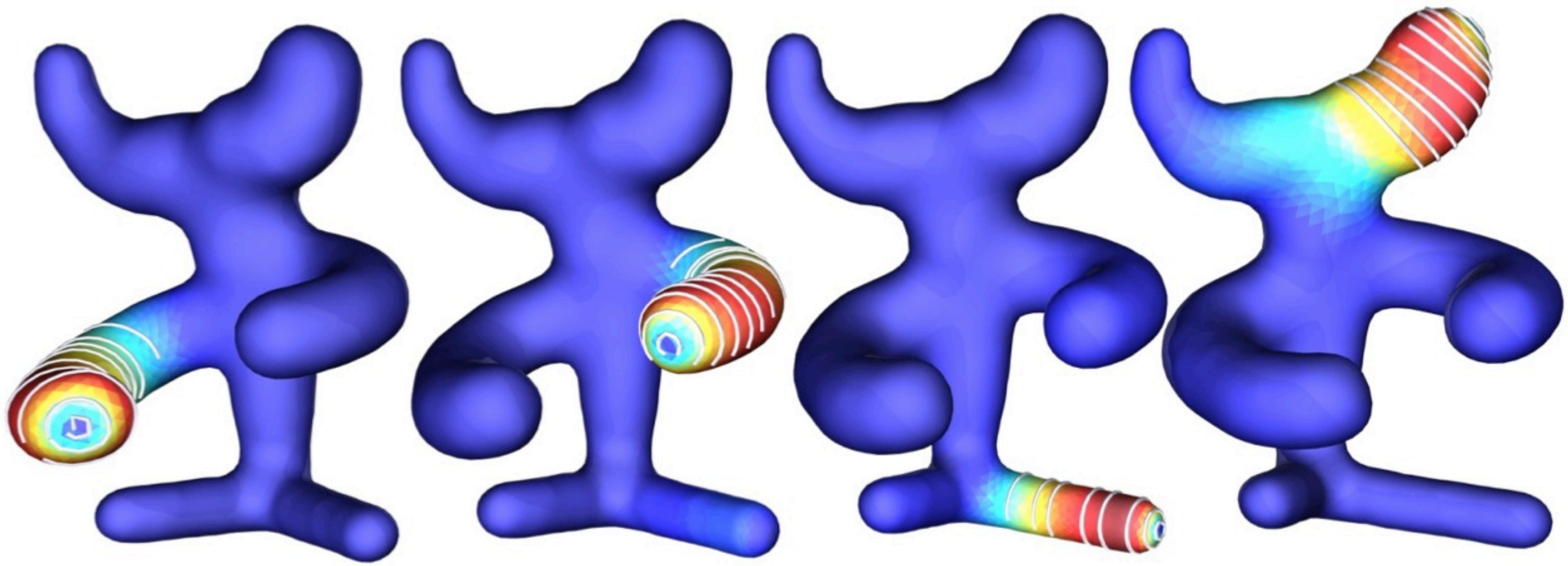
Ben-Chen, Butscher, Solomon, Guibas **On discrete killing vector fields and patterns on surfaces**, SGP 2010

Pattern Generation



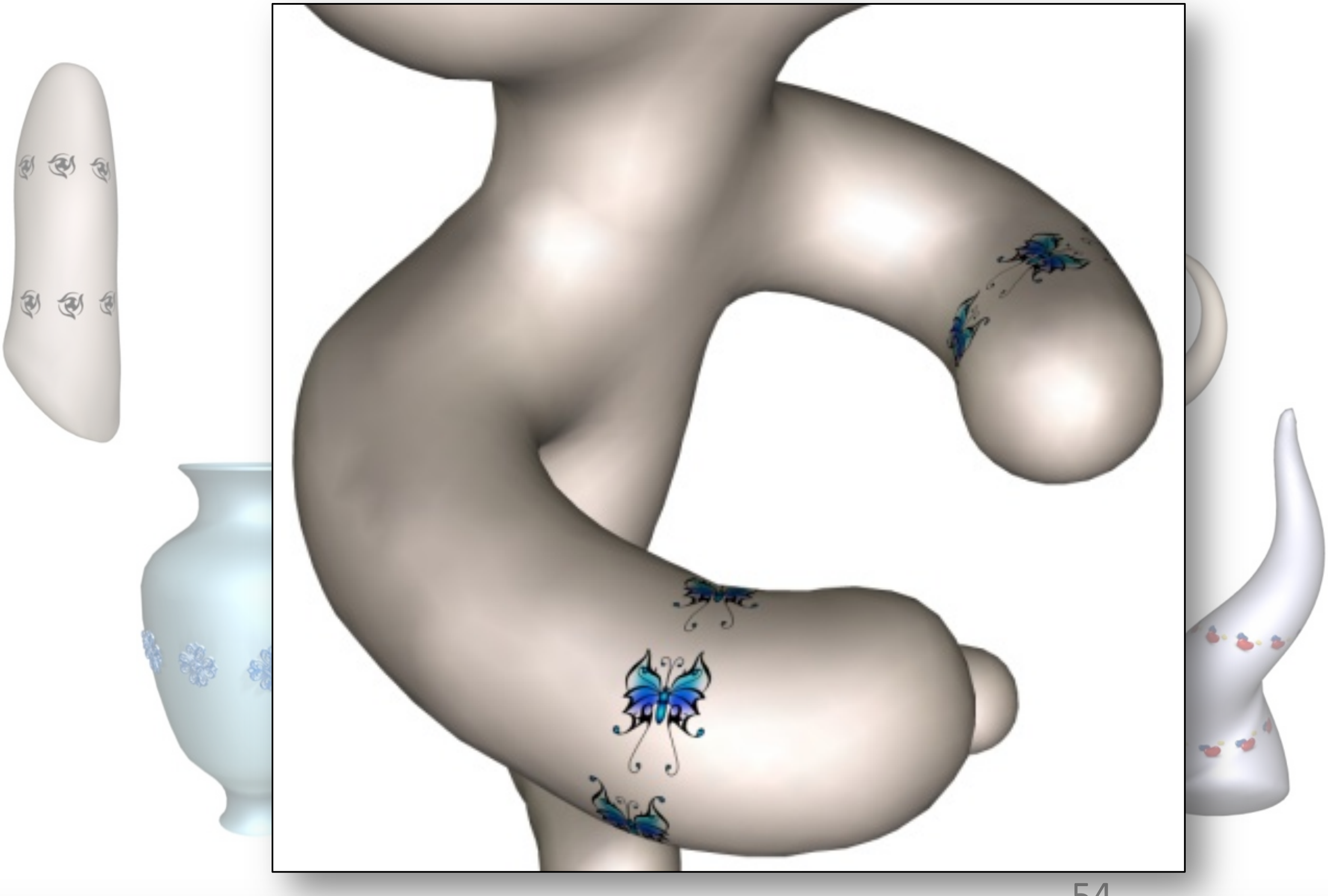


Multiple Continuous Symmetries



First Eigenvector   #2   #3   #4

Pattern Generation



Conclusions

Intrinsic Symmetry Detection:

- Formulated as finding *intrinsic* distance-preserving maps.
- Often solved using isometric matching techniques.
- Theoretically equivalent to extrinsic symmetry detection but in higher dimensional space.
- Continuous symmetries treated with differential methods.

Open problems:

- Good theory for the approximate setting.
- Practical automatic methods.
- Better understanding of the correct deformation space.