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Discrete Laplace operators: No free lunch

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Abstract

Discrete Laplace operators are ubiquitous in applications spanning geometric modeling to simulation. For robustness and efficiency, many applications require discrete operators that retain key structural properties inherent to the continuous setting. Building on the smooth setting, we present a set of natural properties for discrete Laplace operators for triangular surface meshes. We prove an important theoretical limitation: discrete Laplacians cannot satisfy all natural properties; retroactively, this explains the diversity of existing discrete Laplace operators. Finally, we present a family of operators that includes and extends well-known and widely-used operators.

1. Introduction

Discrete Laplace operators on triangular surface meshes span the entire spectrum of geometry processing applications, including mesh filtering, parameterization, pose transfer, segmentation, reconstruction, re-meshing, compression, simulation, and interpolation via barycentric coordinates [Tau00, Zha04, FH05, Sor05].

In applications one often requires certain structural properties of discrete Laplacians—such as symmetry, sparsity, linear precision, positivity, and convergence—requirements that are motivated by an attempt to keep properties of the continuous case, leading to a large and diverse pool of discrete versions. What is missing is a characterization of this vast pool by means of a unified *conceptual treatment*.

As a step toward such a unified treatment, we describe a set of natural properties for discrete Laplace operators on triangular surface meshes (§2). Building on a century-old theorem by Maxwell and Cremona [Max64, Cre90], we prove an important theoretical limitation: not all the natural properties can be satisfied simultaneously, *i.e.*, a 'perfect' discrete Laplacian does not exist (§3). This result imposes a taxonomy on all discrete Laplacians, by considering those properties that they *fail* to respect. Retroactively, this explains the diversity of existing Laplacians proposed in the literature, as different applications are bound to choose different operators. We complement this analysis with a framework for constructing sparse symmetric discrete Laplacians (§4).

1.1. Properties of smooth Laplacians

Consider a smooth surface S, possibly with boundary, equipped with a Riemannian metric, *i.e.*, an intrinsic notion of distance. Let the intrinsic L^2 inner product of functions u and v on S be denoted by $(u,v)_{L^2} = \int_S uv\, dA$, and let $\Delta = -\operatorname{div}\operatorname{grad}$ denote the intrinsic smooth Laplace-Beltrami operator [Ros97]. We list salient properties of this operator:

(NULL) $\Delta u = 0$ whenever u is constant.

(SYM) Symmetry: $(\Delta u, v)_{L^2} = (u, \Delta v)_{L^2}$ whenever u and v are sufficiently smooth and vanish along the boundary of S.

(LOC) Local support: for any pair $p \neq q$ of points, $\Delta u(p)$ is independent of $u(q)$. Altering the function *value* at a distant point will not affect the *action* of the Laplacian locally.

(LIN) Linear precision: $\Delta u = 0$ whenever S is part of the Euclidean plane, and $u = ax + by + c$ is a linear function on the plane.

(MAX) Maximum principle: harmonic functions (those for which $\Delta u = 0$ in the interior of S) have no local maxima (or minima) at interior points.

(PSD) Positive semi-definiteness: the *Dirichlet energy*, $E_D(u) = \int_S \|\operatorname{grad} u\|^2\, dA$, is non-negative. By our choice of sign for Δ, we obtain $E_D(u) = (\Delta u, u)_{L^2} \geq 0$ whenever u is sufficiently smooth and vanishes along the boundary of S.

In applications, one often requires a *discrete* Laplacian having properties corresponding to (some subset of) the properties listed above.

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2. Discrete Laplacians

Discrete Laplacians defined Consider a triangular surface mesh Γ, with vertex set V, edge set E, and face set F. We define a *discrete Laplace operator* on Γ by its linear action on vertex-based functions,

$$(\mathrm{L}u)_i = \sum_j \omega_{ij}(u_i - u_j) \;, \qquad (1)$$

where i and j refer to vertex labels. Note that (1) automatically implies that L satisfies (NULL). Vice-versa, *any* linear operator on function values at vertices, $(\mathrm{L}u)_i = \sum_j l_{ij} u_j$, which vanishes on constants, satisfies $0 = \sum_j l_{ij}$, and can hence be written as in (1) by setting $\omega_{ij} = -l_{ij}$. The properties of L are encoded by the coefficient matrix, (ω_{ij}).

Desired properties for discrete Laplacians We describe a set of natural properties for discrete Laplacians. Each property is primarily motivated by a core structural property of the smooth Laplacian, but where possible we attempt to provide additional geometric and physical intuition.

SYMMETRY (SYM): $\omega_{ij} = \omega_{ji}$. Motivation: Real symmetric matrices exhibit real eigenvalues and orthogonal eigenvectors.

LOCALITY (LOC): Weights are associated to mesh edges (1-ring support), so that $\omega_{ij} = 0$ if i and j do not share an edge in Γ. Changing the *function value* u_j will not alter the Laplacian's *action* $(\mathrm{L}u)_i$, if i and j do not share an edge. Motivation: Smooth Laplacians govern diffusion processes via $u_t = -\Delta u$. When discretized via random walks on graphs, (ω_{ij}) are transition probabilities along *edges* of Γ.

LINEAR PRECISION (LIN): $(\mathrm{L}u)_i = 0$ at each interior vertex whenever Γ is straight-line embedded into the plane and u is a *linear* function on the plane, point-sampled at the vertices of Γ. This is *equivalent* to requiring that

$$0 = (\mathrm{L}\mathbf{x})_i = \sum_j \omega_{ij}(\mathbf{x}_i - \mathbf{x}_j) \qquad (2)$$

for all interior vertex labels i, where $\mathbf{x} \in \mathbb{R}^{2|V|}$ denotes the vector of positions of the $|V|$ vertices of Γ in the plane[†]. Motivation: In graphics applications, (2) is desirable for (*i*) de-noising, where one expects to remove normal noise only but not to introduce tangential vertex drift [DMSB99], (*ii*) parameterization, where one expects planar regions to remain invariant under parameterization [FH05], and (*iii*) plate bending energies, which must vanish for flat configurations [WBH*07].

POSITIVE WEIGHTS (POS): $\omega_{ij} \geq 0$ whenever $i \neq j$. Additionally we require that for each vertex i there exists at least one vertex j such that $\omega_{ij} > 0$. Motivation: (*i*) (POS) is a sufficient condition for a *discrete maximum principle* (recall (MAX) from the smooth case). (*ii*) Physically, in diffusion problems corresponding to $u_t = -\Delta u$, (POS) assures that flow travels from regions of higher to regions of lower potential, not vice-versa. (*iii*) (POS) establishes a connection to *barycentric coordinates* by setting

$$\lambda_{ij} = \frac{\omega_{ij}}{\sum_{j\neq i}\omega_{ij}} \quad \text{so that} \quad \sum_{j\neq i}\lambda_{ij} = 1 \;.$$

Indeed, u is discrete harmonic ($(\mathrm{L}u)_i = 0$ at all interior vertices) if and only if u_i is a convex combination of its neighbors ($u_i = \sum_{j\neq i}\lambda_{ij}u_j$). (*iv*) The combination (LOC)+(LIN)+(POS) is related to Tutte's embedding theorem for planar graphs [Tut63, GGT06]: positive weights associated to edges yield a straight-line embedding of an abstract planar graph. For fixed boundary vertices, this embedding is unique, and it satisfies (LIN) by construction.

POSITIVE SEMI-DEFINITENESS (PSD): L is symmetric positive semi-definite with respect to the standard inner product and has a one-dimensional kernel. Motivation: The *non-negative* discrete Dirichlet energy is given by $E_D(u) = \sum_{i,j}\omega_{ij}(u_i - u_j)^2$. Note that (SYM) and (POS) imply (PSD), but (PSD) does not imply (POS).

CONVERGENCE (CON): $\mathrm{L}_n \to \Delta$, in the sense that solutions to the discrete Dirichlet problem, involving L_n, converge to the solution of the smooth Dirichlet problem, involving Δ, under appropriate refinement conditions and in appropriate norms [HPW07]. Motivation: (CON) is indispensable when seeking to approximate solutions to PDEs.

Examples We briefly survey several Laplacians used in computer graphics. Purely *combinatorial* Laplacians [Zha04], such as the umbrella operator ($\omega_{ij} = 1$ iff vertex i and j share edge) and the Tutte Laplacian, ($\omega_{ij} = 1/d_i$, where d_i denotes the valence of vertex i) fail to be geometric, *i.e.*, they violate (LIN). Floater's *mean value weights* and the Wachspress coordinates are widely used for mesh parameterization [FH05], but violate (SYM) and (CON). The ubiquitous *cotan weights* [PP93] and their variants, commonly used for mesh de-noising, violate (POS) on general meshes.

To resolve cotan's violation of (POS), [BS05] uses the *intrinsic Delaunay* triangulation of the polyhedral surface, at the cost of violating (LOC). One could alter the definition of (LOC) so that it refers to the intrinsic Delaunay triangulation instead of the input mesh, Γ (in general these two triangulations have differing edges). Even so, an extended notion of locality would be violated: there is no universal (input-independent) integer k, such that the Delaunay edges incident to i can be computed from the knowledge *only* of a k-neighborhood of i in Γ. We refer to §3.3 for further discussion, and summarize the situation:

[†] The equivalence follows from observing that (2) implies that L vanishes on two linear functions, the $x-$ and $y-$coordinates. Since L vanishes on constants by definition, it follows that it vanishes on all linear functions.

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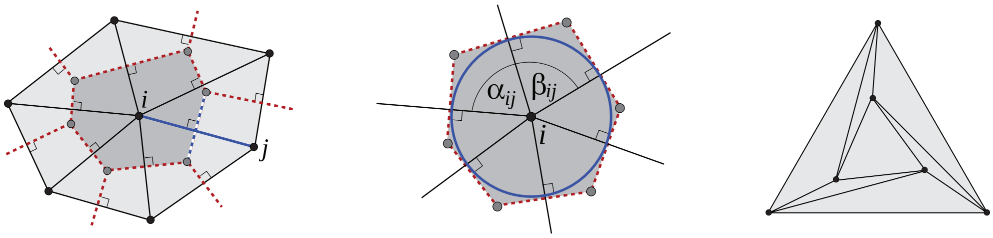

Figure 1: Left*: Primal graph (solid lines) and orthogonal dual (dashed lines), with edge e_{ij} and its dual highlighted. The dark shaded region defines the dual cell, $\star i$.* Middle*: Mean value weights correspond to dual edges tangent to the unit circle around the center vertex.* Right*: The projection of the Schönhardt polytope is not regular, so it does not allow for a discrete Laplacian satisfying* (SYM)+(LOC)+(LIN)+(POS)*.*

	SYM	LOC	LIN	POS	PSD	CON
MEAN VALUE	○	●	●	●	○	○
INTRINSIC DEL	●	○	●	●	●	?
COMBINATORIAL	●	●	○	●	●	○
COTAN	●	●	●	○	●	●

Observe that none of the Laplacians considered in graphics fulfill *all* desired properties. Even more: none of them satisfy the first four properties. This is not a coincidence:

3. No free lunch

Main result Not all meshes admit Laplacians satisfying properties (SYM), (LOC), (LIN), and (POS) simultaneously.

We prove our main result by interpreting a theorem known to Maxwell and Cremona [Max64, Cre90]. Our contribution is to relate their classical result to the study of discrete Laplacians (and barycentric coordinates) in graphics. While the technical tools used here are not new, we use them in developing the *central obstruction* to the existence of 'perfect' discrete Laplacians.

As a first step of deriving this obstruction (§3.1), we establish a correspondence between properties (SYM)+(LOC)+(LIN) and *orthogonal (reciprocal) dual graphs*, based on the Maxwell-Cremona theorem. In a second step (§3.2), we show that orthogonal duals which additionally satisfy (POS) correspond to *regular triangulations*. Since not every mesh is regular, it follows that general meshes do not admit Laplacians that satisfy (SYM)+(LOC)+(LIN)+(POS).

3.1. Geometric Laplacians and orthogonal dual graphs

Maxwell-Cremona view One may view the weights, ω_{ij}, as stresses on a planar framework (with $\omega_{ij} > 0$ corresponding to pulling stresses and $\omega_{ij} < 0$ for pushing stresses). Then (2) is the Euler-Lagrange equation of the equilibrium state of the framework when all boundary vertices are held fixed. The Maxwell-Cremona theorem states that the framework is in equilibrium *if and only if* there exists a orthogonal (reciprocal) dual framework.

Orthogonal duals Consider a planar graph, Γ, *embedded into the plane* with *straight* edges that do not cross. An *orthogonal dual* is a realization of the dual graph, $\Gamma^* = (V^*, E^*, F^*) = (F, E, V)$, in the plane, with straight edges *orthogonal* to primal edges (viewed as vectors in the plane)[‡], see Figure 1-*left*.

To relate orthogonal duals to our properties, first consider a Laplacian on Γ that satisfies (SYM)+(LOC)+(LIN). For each primal edge $\mathbf{e}_{ij}$ of Γ, viewed as a vector in the plane, we can define a corresponding *dual* edge by

$$\star\mathbf{e}_{ij} = \mathrm{R}^{90}(\omega_{ij}\mathbf{e}_{ij}) ,$$

where R^{90} denotes rotation by 90 degrees in the plane. In general, dual edges do not necessarily form closed cycles when moving around an interior primal vertex, *i.e.*, in general, $\sum_j \star\mathbf{e}_{ij} \neq 0$. However, in our case, it is straightforward to check that (2) provides exactly the requisite cycle condition. Therefore, we obtain a realization of the dual graph in the plane whose edges are *orthogonal* to primal edges (viewed as vectors in the plane). Observe that the (straight) edges of Γ^* are allowed to cross because we allow for negative (primal) weights.

Vice versa, consider a pair (Γ, Γ^*) of a primal graph and a corresponding orthogonal dual, both embedded into the plane with straight edges. We obtain weights per primal edge via

$$\omega_{ij} := \frac{|\star\mathbf{e}_{ij}|}{|\mathbf{e}_{ij}|} . \tag{3}$$

Here, $|\mathbf{e}_{ij}|$ denotes the usual Euclidean length, and $|\star\mathbf{e}_{ij}|$ denotes the *signed* Euclidean length of the dual edge. The sign is obtained as follows. The dual edge, $\star\mathbf{e}_{ij}$, connects two dual vertices $\star f_1$ and $\star f_2$, corresponding to the primal faces f_1 and f_2. The sign of $|\star\mathbf{e}_{ij}|$ is *positive* if along the direction of the ray from $\star f_1$ through $\star f_2$, the primal face f_1

[‡] Our definition of *orthogonal duals* is different from the one of [Aur87] who considers what we call *positive* orthogonal duals here.

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lies before f_2. The sign is negative otherwise. With this sign convention, one readily checks that (3) implies (2). We therefore obtain a Laplacian satisfying (SYM)+(LOC)+(LIN).

Examples Discrete Laplacians derived from orthogonal duals on arbitrary (including non-planar) triangular surfaces were recently introduced in [Gli05], however, without noting the *equivalence* to (SYM)+(LOC)+(LIN) in the planar case. A prominent example of orthogonal duals are the cotan weights [PP93], which (as noted in [DHLM05]) arise from assigning dual vertices to *circumcenters* of primal triangles.

If we drop (SYM) from the previous discussion, we still obtain an orthogonal *dual face per primal vertex*, although these dual faces no longer fit into a consistent dual graph. When the dual edges all have positive length, we obtain an operator satisfying (LOC)+(LIN)+(POS) but not (SYM). [FHK06] explored a subspace of this case: a one-parameter family of *linear precision barycentric coordinates*, including mean value and Wachspress coordinates (see Figure 1-*middle*). [LBS06] showed that each member of this family corresponds to a specific choice of orthogonal dual face per primal vertex.

3.2. Positive Laplacians and regular triangulations

We now show the central obstruction: A triangulation of the plane allows for discrete Laplacians which satisfy (SYM)+(LOC)+(LIN)+(POS) if and only if the triangulation is regular.

While there are various equivalent definitions of *regularity* [Ede01], the above obstruction immediately follows when combining the previous discussion with an observation of Aurenhammer [Aur87]: a straight-line triangulation of the plane is regular if and only if it allows for a *positive* orthogonal dual, *i.e.*, a dual with positive weights, ω_{ij}. Unfortunately, an arbitrary input mesh, Γ, is not guaranteed to be regular, see Figure 1-*right*. This completes the proof of our main result: there are no 'perfect' discrete Laplacians for general meshes.

3.3. Discussion

Extended notion of locality To encompass additional possibilities for discrete operators, one could consider extending (LOC) from 1-rings to k-rings for some fixed $k > 1$, *i.e.*, where ω_{ij} is allowed to be non-zero if i and j are no more than k edges apart. Such an extension would accommodate, *e.g.*, methods using higher-order basis functions. The Laplacians provided in [Xu04], based on Loop subdivision bases, use $k = 2$, but they break (SYM) and (POS). We conjecture, but do not prove, that extending (LOC) to $k > 1$ does not remove the fundamental obstruction to a perfect Laplacian.

Regularity-restoring approaches Motivated by [BS05], one could attempt to circumvent the central obstruction to perfect Laplacians by considering an algorithm that first modifies the input (Γ) mesh combinatorics to ensure regularity. One might then modify the definition of (LOC) to refer to the intrinsic triangulation rather than Γ. We discuss this possibility and conjecture that this route violates another notion of locality of the Laplacian, which we call (LOC2): the existence of a universal (mesh-independent) integer k such that the weights ω_{ij} can be computed from the k-neighborhood of i in the original triangulation Γ.

As in the planar picture, one can turn any (non-flat) triangular mesh into a regular one without changing its intrinsic structure by *intrinsic edge flips* [Gli05, FSBS]. After regularity has been restored via intrinsic edge flips, one could redefine (LOC) with respect to the intrinsic triangulation, rather than Γ, to obtain Laplacians satisfying (SYM)+(LOC)+(LIN)+(POS). Unfortunately, for the specific case of an intrinsic Delaunay re-triangulation of Γ, we observed in §2 that (LOC2) would still be violated.

We conjecture that any approach that intrinsically restores regularity must violate (LOC2). Our belief stems from the link between regularity and *weighted* Delaunay triangulation [Ede01]: given a weighted Delaunay triangulation, when a vertex (arbitrarily far away from a given vertex i) is moved, the restoration of the weighted-Delaunay invariants can require re-tessellation or re-assignment of weights locally around i.

3.4. Taxonomy of the literature

In hindsight, our result explains the diversity of discrete Laplacians considered in graphics, each application choosing the subset of properties closest tailored to their needs: dropping (LOC) yields intrinsic (weighted) Delaunay (or meshless) Laplacians, dropping (SYM) gives rise to barycentric coordinates, dropping (LIN) yields combinatorial Laplacians, and dropping (POS) gives rise to cotan weights and their generalization (3).

4. General construction for discrete Laplacians

In this final section, we offer a framework for constructing discrete Laplacians using adjoint operators and L^2 inner products. We show that (SYM) and (LOC) arise from choosing *diagonal* inner products, (LOC2) holds if inner products depend only on local k-neighborhoods of Γ, (POS) corresponds to inner products with *positive* entries, (PSD) arises from *positive semi-definite* inner products, and (LIN) corresponds to a *geometric* choice.

Construction It is known from the continuous setting that the Laplacian on functions can be written as $\Delta = \delta \mathrm{d} u$, where d denotes the usual *metric-free* derivative taking 0-forms (functions) to 1-forms, and δ is the *adjoint operator*, taking 1-forms to 0-forms. Using L^2 inner products, δ is defined by the identity $(\mathrm{d}u, \alpha)_{L^2_1} = (u, \delta\alpha)_{L^2_0}$, where u is a function and α is a 1-form. Notice that d is defined independent of any metric, whereas δ cannot be defined without a metric. For

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the Laplacian we obtain

$$(\Delta u, v)_{L_0^2} = (\delta \mathrm{d}u, v)_{L_0^2} = (\mathrm{d}u, \mathrm{d}v)_{L_1^2} \,. \tag{4}$$

In the discrete case, we identify 0-forms with values at vertices, and 1-forms with values at edges. The metric-independent derivative, d, taking 0-forms to 1-forms is

$$(\mathrm{d}u)(e_{ij}) = u_j - u_i \,.$$

It remains to define the adjoint operator, δ. As before, its definition is metric-dependent. Denoting edge lengths by $|e|$, we obtain L^2 inner products for 0-forms and 1-forms by summing over all vertex pairs (j, j'), respectively all edge pairs (e, e'):

$$(u,v)_{L_0^2} = \sum_{j,j'} m_{jj'} u_j v_{j'} \quad \text{and} \quad (\alpha,\beta)_{L_1^2} = \sum_{e,e'} l_{ee'} \frac{\alpha(e)}{|e|} \frac{\beta(e')}{|e'|} \,.$$

Notice that the square matrix $(m_{jj'})$ is vertex-based, while the square matrix $(l_{ee'})$ is edge-based. In the specific case of *diagonal* matrices, we can treat $(l_{ee'})$ as vertex-based by setting $l_{ij} := l_{e_{ij}e_{ij}}$. From (4) we obtain

$$(\mathrm{L}u)_i := (\Delta u, 1_i)_{L_0^2} = m_{ii}(\Delta u)_i = \sum_j \frac{l_{ij}}{|e_{ij}|^2} \left(u_i - u_j\right) \,, \tag{5}$$

where 1_i is the discrete Dirac delta function, which has unit value at vertex i and vanishes on all others. Observe that by appropriate choice of inner products, l_{ij}, we *recover all discrete Laplacians* (1) which satisfy (LOC) and (SYM).

Properties Observe that (LOC) and (SYM) are satisfied automatically in (5), (LOC2) holds if l_{ij} can be computed from local mesh information, (POS) is equivalent to $l_{ij} \geq 0$, and (PSD) is equivalent to $(\mathrm{d}u, \mathrm{d}u)_{L_1^2} \geq 0$ with equality only if u is constant. Finally, (LIN) corresponds to geometric inner products. To see this, recall from §3.1 that (LIN) corresponds to orthogonal duals. The geometric view is obtained by setting $m_{ii} = |\star i|$ (*area* of the dual cell), and $l_{ij} = |\star \mathbf{e}_{ij}||\mathbf{e}_{ij}|$ (where $|\star \mathbf{e}_{ij}|$ is signed length), exactly reproducing the weights of (3).

As a concluding remark we note that our inner product view generalizes the approach of [DHLM05], which constructs δ and Δ from a discrete Hodge star, based on circumcentric duals. Indeed, while it is straightforward to generalize the Hodge star framework of [DHLM05] from circumcentric to arbitrary orthogonal duals, it is not obvious whether this approach generalizes to Laplacians not arising from a dual mesh. In contrast, our inner product view is entirely primal-based, with the use of a dual mesh restricted to a special (geometric) case.

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