This supplemental document first describes the derivation of the variance $V[\langle F\rangle_{\text{tsr}}]$ for the (one-sample) two-stage resampling estimator in Eq. (12) of the main paper. Then the derivation of the variance $V[\langle I_t\rangle^N_{\text{tsr}}]$ for the (N-sample) two-stage resampling estimator for resampling strategies t is described. Finally, we derive the weighting function that aims to reduce the variance $V[\langle I_t\rangle^N_{\text{tsr}}]$.

1. Derivation of variance $V[\langle F\rangle_{\text{tsr}}]$

To derive the variance of (one-sample) two-stage resampling estimator $V[\langle F\rangle_{\text{tsr}}] = E[\langle F^2\rangle_{\text{tsr}}] - E[\langle F\rangle_{\text{tsr}}]^2$, we first derive the expected value $E[\langle F\rangle_{\text{tsr}}]$ and then the second moment $E[\langle F^2\rangle_{\text{tsr}}]$ is described. Let $\mathbf{X}$ be the set of M_1 proposals $\mathbf{X} = \{X_1,\ldots,X_{M_1}\}$ sampled by the sampling pdf p, and $\bar{\mathsf{X}}$ be the subset of $\mathbf{X}$ comprising of M_2 proposals $\bar{\mathsf{X}} = \{X_{i_1},\ldots,X_{i_{M_2}}\}$. The expected value $E[\langle F\rangle_{\text{tsr}}]$ is calculated by:

$$E[\langle F\rangle_{\text{tsr}}] = E\left[\frac{f(X)}{\hat{q}_2(X)}\left(\frac{1}{M_1}\sum_{j=1}^{M_1}\frac{\hat{q}_1(X_j)}{p(X_j)}\right)\left(\frac{1}{M_2}\sum_{j=1}^{M_2}\frac{\hat{q}_2(X_{i_j})}{\hat{q}_1(X_{i_j})}\right)\right]$$

By using $E[abc] = E[E[E[a|b]b|c]c]$, the expected value $E[\langle F\rangle_{\text{tsr}}]$ is represented as:

$$E\left[E\left[E\left[\frac{f(X)}{\hat{q}_2(X)}|\bar{\mathsf{X}}\right]\left(\frac{1}{M_2}\sum_{j=1}^{M_2}\frac{\hat{q}_2(X_{i_j})}{\hat{q}_1(X_{i_j})}\right)|\mathbf{X}\right]\left(\frac{1}{M_1}\sum_{j=1}^{M_1}\frac{\hat{q}_1(X_j)}{p(X_j)}\right)\right].$$

The innermost expected value is calculated by summing the product of $f/\hat{q}_2$ and the pmf over all the proposals in $\bar{\mathsf{X}}$:

$$\begin{aligned}E\left[\frac{f(X)}{\hat{q}_2(X)}|\bar{\mathsf{X}}\right] &= \sum_{j=1}^{M_2}\frac{f(X_{i_j})}{\hat{q}_2(X_{i_j})}\frac{\hat{q}_2(X_{i_j})/\hat{q}_1(X_{i_j})}{\sum_{k=1}^{M_2}\hat{q}_2(X_{i_k})/\hat{q}_1(X_{i_k})}\\ &= \left(\sum_{j=1}^{M_2}\frac{f(X_{i_j})}{\hat{q}_1(X_{i_j})}\right)\left(\sum_{k=1}^{M_2}\frac{\hat{q}_2(X_{i_k})}{\hat{q}_1(X_{i_k})}\right)^{-1}.\end{aligned}$$

By substituting this, $E[\langle F\rangle_{\text{tsr}}]$ is expressed as:

$$\begin{aligned}&E\left[E\left[\frac{1}{M_2}\left(\sum_{j=1}^{M_2}\frac{f(X_{i_j})}{\hat{q}_1(X_{i_j})}\right)|\mathbf{X}\right]\left(\frac{1}{M_1}\sum_{j=1}^{M_1}\frac{\hat{q}_1(X_j)}{p(X_j)}\right)\right],\\ &= E\left[E\left[\frac{f(X_{i_1})}{\hat{q}_1(X_{i_1})}|\mathbf{X}\right]\left(\frac{1}{M_1}\sum_{j=1}^{M_1}\frac{\hat{q}_1(X_j)}{p(X_j)}\right)\right].\end{aligned}$$

The inner expected value is calculated by summing the product of $f/\hat{q}_1$ and the pmf over all the proposals in $\mathbf{X}$:

$$\begin{aligned}E\left[\frac{f(X_{i_1})}{\hat{q}_1(X_{i_1})}|\mathbf{X}\right] &= \sum_{i=1}^{M_1}\frac{f(X_i)}{\hat{q}_1(X_i)}\frac{\hat{q}_1(X_i)/p(X_i)}{\sum_{k=1}^{M_1}\hat{q}_1(X_k)/p(X_k)}\\ &= \left(\sum_{i=1}^{M_1}\frac{f(X_i)}{p(X_i)}\right)\left(\sum_{k=1}^{M_1}\frac{\hat{q}_1(X_k)}{p(X_k)}\right)^{-1}.\end{aligned}$$

By substituting this, the expected value $E[\langle F\rangle_{\text{tsr}}]$ is simplified to:

$$E\left[\frac{1}{M_1}\sum_{i=1}^{M_1}\frac{f(X_i)}{p(X_i)}\right] = E\left[\frac{f(X)}{p(X)}\right]. \tag{1}$$

Next, the second moment $E[\langle F^2\rangle_{\text{tsr}}]$ is calculated as:

$$E\left[E\left[E\left[\left(\frac{f(X)}{\hat{q}_2(X)}\right)^2|\bar{\mathsf{X}}\right]\left(\frac{1}{M_2}\sum_{j=1}^{M_2}\frac{\hat{q}_2(X_{i_j})}{\hat{q}_1(X_{i_j})}\right)^2|\mathbf{X}\right]\left(\frac{1}{M_1}\sum_{j=1}^{M_1}\frac{\hat{q}_1(X_j)}{p(X_j)}\right)^2\right].$$

We first expand the innermost expected value as:

$$\begin{aligned}E\left[\left(\frac{f(X)}{\hat{q}_2(X)}\right)^2|\bar{\mathsf{X}}\right] &= \sum_{j=1}^{M_2}\frac{f^2(X_{i_j})}{\hat{q}_2^2(X_{i_j})}\frac{\hat{q}_2(X_{i_j})/\hat{q}_1(X_{i_j})}{\sum_{k=1}^{M_2}\hat{q}_2(X_{i_k})/\hat{q}_1(X_{i_k})}\\ &= \left(\sum_{j=1}^{M_2}\frac{f^2(X_{i_j})}{\hat{q}_1(X_{i_j})\hat{q}_2(X_{i_j})}\right)\left(\sum_{k=1}^{M_2}\frac{\hat{q}_2(X_{i_k})}{\hat{q}_1(X_{i_k})}\right)^{-1}.\end{aligned}$$

By substituting this, the inner expected value is transformed as:

$$
\begin{aligned}
&E\left[\left(\frac{1}{M_2}\sum_{j=1}^{M_2}\frac{f^2(X_{i_j})}{\hat{q}_1(X_{i_j})\hat{q}_2(X_{i_j})}\right)\left(\frac{1}{M_2}\sum_{j=1}^{M_2}\frac{\hat{q}_2(X_{i_j})}{\hat{q}_1(X_{i_j})}\right)|\mathbf{X}\right] \\
&=\frac{1}{M_2}E\left[\frac{f^2}{\hat{q}_1^2}|\mathbf{X}\right]+\left(1-\frac{1}{M_2}\right)E\left[\frac{f^2}{\hat{q}_1\hat{q}_2}|\mathbf{X}\right]E\left[\frac{\hat{q}_2}{\hat{q}_1}|\mathbf{X}\right],
\end{aligned}
$$

These expected values are expanded as:

$$
\begin{aligned}
E\left[\frac{f^2(X_{i_1})}{\hat{q}_1^2(X_{i_1})}|\mathbf{X}\right] &= \left(\sum_{j=1}^{M_1}\frac{f^2(X_j)}{\hat{q}_1(X_j)p(X_j)}\right)\left(\sum_{k=1}^{M_1}\frac{\hat{q}_1(X_k)}{p(X_k)}\right)^{-1} \\
E\left[\frac{f^2(X_{i_1})}{\hat{q}_1(X_{i_1})\hat{q}_2(X_{i_1})}|\mathbf{X}\right] &= \left(\sum_{j=1}^{M_1}\frac{f^2(X_j)}{\hat{q}_2(X_j)p(X_j)}\right)\left(\sum_{k=1}^{M_1}\frac{\hat{q}_1(X_k)}{p(X_k)}\right)^{-1} \\
E\left[\frac{\hat{q}_2(X_{i_1})}{\hat{q}_1(X_{i_1})}|\mathbf{X}\right] &= \left(\sum_{j=1}^{M_1}\frac{\hat{q}_2(X_j)}{p(X_j)}\right)\left(\sum_{k=1}^{M_1}\frac{\hat{q}_1(X_k)}{p(X_k)}\right)^{-1}
\end{aligned}
$$

By substituting these, the outermost expected value is expressed as:

$$
\begin{aligned}
&\frac{1}{M_2}E\left[\left(\frac{1}{M_1}\sum_{j=1}^{M_1}\frac{f^2(X_j)}{\hat{q}_1(X_j)p(X_j)}\right)\left(\frac{1}{M_1}\sum_{k=1}^{M_1}\frac{\hat{q}_1(X_k)}{p(X_k)}\right)\right] \\
&+\left(1-\frac{1}{M_2}\right)E\left[\left(\frac{1}{M_1}\sum_{j=1}^{M_1}\frac{f^2(X_j)}{\hat{q}_2(X_j)p(X_j)}\right)\left(\frac{1}{M_1}\sum_{k=1}^{M_1}\frac{\hat{q}_2(X_k)}{p(X_k)}\right)\right] \\
&=\frac{1}{M_1}E\left[\frac{f^2(X)}{p^2(X)}\right]+\left(1-\frac{1}{M_1}\right)\frac{1}{M_2}E\left[\frac{f^2(X)}{\hat{q}_1(X)p(X)}\right]E\left[\frac{\hat{q}_1(X)}{p(X)}\right] \\
&+\left(1-\frac{1}{M_1}\right)\left(1-\frac{1}{M_2}\right)E\left[\frac{f^2(X)}{\hat{q}_2(X)p(X)}\right]E\left[\frac{\hat{q}_2(X)}{p(X)}\right].
\end{aligned}
$$

We omit the augments from here on for the brevity. Then the product of two expected values $E[f^2/\hat{q}_1 p]E[\hat{q}_1/p]$ is rewritten as:

$$
\begin{aligned}
E\left[\frac{f^2}{\hat{q}_1 p}\right]E\left[\frac{\hat{q}_1}{p}\right] &= \int\frac{f^2(x)}{\hat{q}_1(x)}dx\cdot\int\hat{q}_1(x)dx \\
&=\int\frac{f^2(x)}{\hat{q}_1(x)/\int\hat{q}_1(x)dx}dx=\int\frac{f^2(x)}{q_1(x)}dx=E\left[\frac{f^2}{q_1^2}\right],
\end{aligned}
$$

where q_1 is the target pdf of the target distribution $\hat{q}_1$. Analogously, the product of two expected values $E[f^2/\hat{q}_2 p]E[\hat{q}_2/p]$ is rewritten by using the target pdf q_2 as:

$$
E\left[\frac{f^2}{\hat{q}_2 p}\right]E\left[\frac{\hat{q}_2}{p}\right]=E\left[\frac{f^2}{q_2^2}\right].
$$

By substituting these, $E[\langle F^2\rangle_{\text{tsr}}]$ is expressed as:

$$
\frac{1}{M_1}E\left[\frac{f^2}{p^2}\right]+\left(1-\frac{1}{M_1}\right)\left(\frac{1}{M_2}E\left[\frac{f^2}{q_1^2}\right]+\left(1-\frac{1}{M_2}\right)E\left[\frac{f^2}{q_2^2}\right]\right).
$$

Finally, the variance $V[\langle F\rangle_{\text{tsr}}]=E[\langle F^2\rangle_{\text{tsr}}]-E[\langle F\rangle_{\text{tsr}}]^2$ is calculated by subtracting the square of the expected value from the second moment. Here, the expected value can be rewritten by using the target pdfs q_1 and q_2 as $E[f/p]=\int f(x)dx=E[f/q_1]=E[f/q_2]$, and the sum of the coefficients in the second moment is equal to one (i.e., $1/M_1+(1-M_1)(1/M_2+(1-1/M_2))=1$). Then the square of the expected value $E[f/p]^2$ can be expressed as:

$$
\frac{1}{M_1}E\left[\frac{f}{p}\right]^2+\left(1-\frac{1}{M_1}\right)\left(\frac{1}{M_2}E\left[\frac{f}{q_1}\right]^2+\left(1-\frac{1}{M_2}\right)E\left[\frac{f}{q_2}\right]^2\right).
$$

By subtracting these from the second moment, the variance is represented by Eq. (12) of the main paper.

2. Derivation of Variance for Two-Stage Resampling Estimator

In this supplemental material, we explain the derivation of the variance for (N-sample) two-stage resampling estimator $V[\langle I_t\rangle^N_{\text{tsr}}]$. As described in the paper, we assume that the importance sampling is used (instead of the stratified sampling) in the first resampling stage during the derivation of the variance. This is a reasonable assumption since the variance of the stratified sampling is smaller than that of the importance sampling in general. By using this assumption, the two-stage resampling estimator $V[\langle I_t\rangle^N_{\text{tsr}}]$ for a given eye sub-path sample $\bar{Z}_t$ is calculated from:

$$\langle I_t\rangle^N_{\text{tsr}} = \frac{1}{M_1 M_2 N}\sum_{k=1}^{N}\frac{w_t(\bar{Y}_k\bar{Z}_t)f(\bar{Y}_k\bar{Z}_t)}{p(\bar{Y}_k)p(\bar{Z}_t)P_r(\bar{Y}_k|\bar{\mathbf{Y}})},$$

where M_1 is the number of tracing light sub-paths, $\bar{Y}$ is the set of pre-sampled light sub-paths as $\bar{\mathbf{Y}} = \{\bar{Y}_{1,1},\ldots,\bar{Y}_{s,i},\ldots\}$ where $\bar{Y}_{s,i}$ is i-th light sub-path sample with s vertices. M_2 is the number of light sub-paths for the second resampling stage, and the subset of pre-sampled light sub-paths used in the second resampling stage is represented by $\bar{\mathsf{Y}} = \{\bar{Y}_{i_1},\ldots,\bar{Y}_{i_j},\ldots,\bar{Y}_{i_{M_2}}\}$ where $\bar{Y}_{i_j}\in\bar{\mathbf{Y}}$. $\bar{Y}_k$ is the k-th sample resampled from the subset. To simplify the notation, we concatenate the light sub-path with the sampled eye sub-path $\bar{Z}_t$, and the light sub-paths such as $\bar{Y}_k$, $\bar{Y}_{s,i}$, and $\bar{Y}_{i_j}$ are represented by the full light path samples as $\bar{X}_k = \bar{Y}_k\bar{Z}_t$, $\bar{X}_{s,i} = \bar{Y}_{s,i}\bar{Z}_t$, and $\bar{X}_{i_j} = \bar{Y}_{i_j}\bar{Z}_t$. We also represent the measurement contribution function f weighted by the weighting function w_t with $f_t(\bar{x}) \equiv w_t(\bar{x})f(\bar{x})$, and the set $\bar{\mathbf{Y}}$ is replaced with $\bar{\mathbf{X}} = \{\bar{X}_{1,1},\ldots,\bar{X}_{s,i},\ldots\}$. The subset used in the second resampling stage is denoted by $\bar{\mathsf{X}} = \{\bar{X}_{i_1},\ldots,\bar{X}_{i_{M_2}}\}$. By using these notations, the two-stage resampling estimator $\langle I_t\rangle^N_{\text{tsr}}$ is simplified to:

$$\langle I_t\rangle^N_{\text{tsr}} = \frac{1}{M_1 M_2 N}\sum_{k=1}^{N}\frac{f_t(\bar{X}_k)}{p(\bar{X}_k)P_r(\bar{X}_k|\bar{\mathbf{X}})}, \quad (2)$$

The resampling pmf P_r is the product of two pmfs as:

$$P_1(\bar{X}_k|\bar{\mathbf{X}}) = \frac{\hat{q}_1(\bar{X}_k)/p(\bar{X}_k)}{\sum_{s\geq 1}\sum_{i=1}^{M_1}\hat{q}_1(\bar{X}_{s,i})/p(\bar{X}_{s,i})}, \quad (3)$$

$$P_2(\bar{X}_k|\bar{\mathsf{X}}) = \frac{\hat{q}_2(\bar{X}_k)/\hat{q}_1(\bar{X}_k)}{\sum_{j=1}^{M_2}\hat{q}_2(\bar{X}_{i_j})/\hat{q}_1(\bar{X}_{i_j})}, \quad (4)$$

where $\hat{q}_1$ and $\hat{q}_2$ are the target distributions for the first resampling stage and the second resampling stage, respectively. By substituting P_1 and P_2 for P_r, $\langle I_t\rangle^N_{\text{tsr}}$ is expressed by:

$$\begin{aligned}\langle I_t\rangle^N_{\text{tsr}} &= \frac{1}{M_1 M_2 N}\sum_{k=1}^{N}\frac{f_t(\bar{X}_k)}{p(\bar{X}_k)P_1(\bar{X}_k|\bar{\mathbf{X}})P_2(\bar{X}_k|\bar{\mathsf{X}})}\\ &= \left(\frac{1}{N}\sum_{k=1}^{N}\frac{f_t(\bar{X}_k)}{\hat{q}_2(\bar{X}_k)}\right)\left(\frac{1}{M_1}\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{\hat{q}_1(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)\left(\frac{1}{M_2}\sum_{j=1}^{M_2}\frac{\hat{q}_2(\bar{X}_{i_j})}{\hat{q}_1(\bar{X}_{i_j})}\right)\end{aligned} \quad (5)$$

Similary, $\langle I_t^2\rangle^N_{\text{tsr}}$ used for the second moment $E[\langle I_t^2\rangle^N_{\text{tsr}}|\bar{Z}_t]$ is represented by:

$$\langle I_t^2\rangle^N_{\text{tsr}} = \left(\frac{1}{N}\sum_{k=1}^{N}\frac{f_t(\bar{X}_k)}{\hat{q}_2(\bar{X}_k)}\right)^2\left(\frac{1}{M_1}\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{\hat{q}_1(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)^2\left(\frac{1}{M_2}\sum_{j=1}^{M_2}\frac{\hat{q}_2(\bar{X}_{i_j})}{\hat{q}_1(\bar{X}_{i_j})}\right)^2. \quad (6)$$

To compute the conditional variance $V[\langle I_t\rangle^N_{\text{tsr}}|\bar{Z}_t] = E[\langle I_t^2\rangle^N_{\text{tsr}}|\bar{Z}_t] - E[\langle I_t\rangle^N_{\text{tsr}}|\bar{Z}_t]^2$, we first explain the derivation of $E[\langle I_t\rangle^N_{\text{tsr}}|\bar{Z}_t]$, then the derivation of $E[\langle I_t^2\rangle^N_{\text{tsr}}|\bar{Z}_t]$ is described.

2.1. Derivation of $E[\langle I_t\rangle^N_{\text{tsr}}|\bar{Z}_t]$

The conditional expected value $E[\langle I_t\rangle^N_{\text{tsr}}|\bar{Z}_t]$ is expressed by:

$$E\left[\left(\frac{1}{N}\sum_{k=1}^{N}\frac{f_t(\bar{X}_k)}{\hat{q}_2(\bar{X}_k)}\right)\left(\frac{1}{M_1}\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{\hat{q}_1(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)\left(\frac{1}{M_2}\sum_{j=1}^{M_2}\frac{\hat{q}_2(\bar{X}_{i_j})}{\hat{q}_1(\bar{X}_{i_j})}\right)|\bar{Z}_t\right]. \quad (7)$$

By using $E[abc] = E[E[E[a|b]b|c]c]$, $E[\langle I_t\rangle^N_{\text{tsr}}|\bar{Z}_t]$ is transformed into:

$$E\left[E\left[E\left[\left(\frac{1}{N}\sum_{k=1}^{N}\frac{f_t(\bar{X}_k)}{\hat{q}_2(\bar{X}_k)}\right)|\bar{\mathsf{X}}\right]\left(\frac{1}{M_2}\sum_{j=1}^{M_2}\frac{\hat{q}_2(\bar{X}_{i_j})}{\hat{q}_1(\bar{X}_{i_j})}\right)|\bar{\mathbf{X}}\right]\left(\frac{1}{M_1}\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{\hat{q}_1(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)|\bar{Z}_t\right].$$

We first expand the innermost expected value as follows:

$$E\left[\left(\frac{1}{N}\sum_{k=1}^{N}\frac{f_t(\bar{X}_k)}{\hat{q}_2(\bar{X}_k)}\right)|\bar{\mathsf{X}}\right]=E\left[\frac{f_t(\bar{X}_1)}{\hat{q}_2(\bar{X}_1)}|\bar{\mathsf{X}}\right]=\sum_{j=1}^{M_2}\frac{f_t(\bar{X}_{i_j})}{\hat{q}_2(\bar{X}_{i_j})}\frac{\hat{q}_2(\bar{X}_{i_j})/\hat{q}_1(\bar{X}_{i_j})}{\sum_{k=1}^{M_2}\hat{q}_2(\bar{X}_{i_k})/\hat{q}_1(\bar{X}_{i_k})}=\left(\sum_{j=1}^{M_2}\frac{f_t(\bar{X}_{i_j})}{\hat{q}_1(\bar{X}_{i_j})}\right)\left(\sum_{j=1}^{M_2}\frac{\hat{q}_2(\bar{X}_{i_j})}{\hat{q}_1(\bar{X}_{i_j})}\right)^{-1} \tag{8}$$

Substituting the above equation cancels out the summation and leads to:

$$E\left[E\left[\left(\frac{1}{M_2}\sum_{j=1}^{M_2}\frac{f_t(\bar{X}_{i_j})}{\hat{q}_1(\bar{X}_{i_j})}\right)|\bar{\mathbf{X}}\right]\left(\frac{1}{M_1}\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{\hat{q}_1(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)|\bar{Z}_t\right]=E\left[E\left[\frac{f_t(\bar{X}_{i_1})}{\hat{q}_1(\bar{X}_{i_1})}|\bar{\mathbf{X}}\right]\left(\frac{1}{M_1}\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{\hat{q}_1(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)|\bar{Z}_t\right] \tag{9}$$

The inner expected value of the above equation is calculated by summing over all proposals in $\bar{\mathbf{X}}$ as:

$$E\left[\frac{f_t(\bar{X}_{i_1})}{\hat{q}_1(\bar{X}_{i_1})}|\bar{\mathbf{X}}\right]=\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{f_t(\bar{X}_{s,i})}{\hat{q}_1(\bar{X}_{s,i})}\cdot\left(\frac{\hat{q}_1(\bar{X}_{s,i})/p(\bar{X}_{s,i})}{\sum_{s\geq 1}\sum_{i=1}^{M_1}\hat{q}_1(\bar{X}_{s,i})/p(\bar{X}_{s,i})}\right)=\left(\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{f_t(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)\left(\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{\hat{q}_1(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)^{-1}$$

By substituting the above equation and canceling out the summation, $E[\langle I_t\rangle_{\mathrm{tsr}}^N|\bar{Z}_t]$ is expressed by:

$$E[\langle I_t\rangle_{\mathrm{tsr}}^N|\bar{Z}_t]=E\left[\frac{1}{M_1}\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{f_t(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}|\bar{Z}_t\right]=\sum_{s\geq 1}E\left[\frac{1}{M_1}\sum_{i=1}^{M_1}\frac{f_t(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}|\bar{Z}_t\right]=\sum_{s\geq 1}E\left[\frac{f_t(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}|\bar{Z}_t\right]. \tag{10}$$

2.2. Derivation of the second moment $E[\langle I_t^2\rangle_{\mathrm{tsr}}^N|\bar{Z}_t]$

Similar to the derivation of $E[\langle I_t\rangle_{\mathrm{tsr}}^N|\bar{Z}_t]$, by using $E[abc]=E[E[E[a|b]b|c]c]$, the conditional expected value $E[\langle I_t^2\rangle_{\mathrm{tsr}}^N|\bar{Z}_t]$ is transformed into:

$$E\left[E\left[E\left[\left(\frac{1}{N}\sum_{k=1}^{N}\frac{f_t(\bar{X}_k)}{\hat{q}_2(\bar{X}_k)}\right)^2|\bar{\mathsf{X}}\right]\left(\frac{1}{M_2}\sum_{j=1}^{M_2}\frac{\hat{q}_2(\bar{X}_{i_j})}{\hat{q}_1(\bar{X}_{i_j})}\right)^2|\bar{\mathbf{X}}\right]\left(\frac{1}{M_1}\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{\hat{q}_1(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)^2|\bar{Z}_t\right].$$

By using the following identity:

$$E\left[\left(\frac{1}{N}\sum_{i=1}^{N}f(X_i)\right)\left(\frac{1}{N}\sum_{i=1}^{N}g(X_i)\right)\right]=\frac{1}{N}E[f(X_1)g(X_1)]+\left(1-\frac{1}{N}\right)E[f(X_1)]E[g(X_1)],$$

the innermost expected value is expressed by:

$$\begin{aligned}
E\left[\left(\frac{1}{N}\sum_{k=1}^{N}\frac{f_t(\bar{X}_k)}{\hat{q}_2(\bar{X}_k)}\right)^2|\bar{\mathsf{X}}\right]&=\frac{1}{N}E\left[\frac{f_t^2(\bar{X}_1)}{\hat{q}_2^2(\bar{X}_1)}|\bar{\mathsf{X}}\right]+\left(1-\frac{1}{N}\right)E\left[\frac{f_t(\bar{X}_1)}{\hat{q}_2(\bar{X}_1)}|\bar{\mathsf{X}}\right]^2\\
&=\frac{1}{N}\sum_{j=1}^{M_2}\frac{f_t^2(\bar{X}_{i_j})}{\hat{q}_2^2(\bar{X}_{i_j})}\frac{\hat{q}_2(\bar{X}_{i_j})/\hat{q}_1(\bar{X}_{i_j})}{\sum_{k=1}^{M_2}\hat{q}_2(\bar{X}_{i_k})/\hat{q}_1(\bar{X}_{i_k})}+\left(1-\frac{1}{N}\right)\left(\sum_{j=1}^{M_2}\frac{f_t(\bar{X}_{i_j})}{\hat{q}_2(\bar{X}_{i_j})}\frac{\hat{q}_2(\bar{X}_{i_j})/\hat{q}_1(\bar{X}_{i_j})}{\sum_{k=1}^{M_2}\hat{q}_2(\bar{X}_{i_k})/\hat{q}_1(\bar{X}_{i_k})}\right)^2\\
&=\frac{1}{N}\left(\sum_{j=1}^{M_2}\frac{f_t^2(\bar{X}_{i_j})}{\hat{q}_1(\bar{X}_{i_j})\hat{q}_2(\bar{X}_{i_j})}\right)\left(\sum_{j=1}^{M_2}\frac{\hat{q}_2(\bar{X}_{i_j})}{\hat{q}_1(\bar{X}_{i_j})}\right)^{-1}+\left(1-\frac{1}{N}\right)\left(\sum_{j=1}^{M_2}\frac{f_t(\bar{X}_{i_j})}{\hat{q}_1(\bar{X}_{i_j})}\right)^2\left(\sum_{j=1}^{M_2}\frac{\hat{q}_2(\bar{X}_{i_j})}{\hat{q}_1(\bar{X}_{i_j})}\right)^{-2}.
\end{aligned}$$

By substituting this, the inner expected value leads to:

$$\begin{aligned}
&E\left[\frac{1}{N}\left(\frac{1}{M_2}\sum_{j=1}^{M_2}\frac{f_t^2(\bar{X}_{i_j})}{\hat{q}_1(\bar{X}_{i_j})\hat{q}_2(\bar{X}_{i_j})}\right)\left(\frac{1}{M_2}\sum_{j=1}^{M_2}\frac{\hat{q}_2(\bar{X}_{i_j})}{\hat{q}_1(\bar{X}_{i_j})}\right)+\left(1-\frac{1}{N}\right)\left(\frac{1}{M_2}\sum_{j=1}^{M_2}\frac{f_t(\bar{X}_{i_j})}{\hat{q}_1(\bar{X}_{i_j})}\right)^2|\bar{\mathbf{X}}\right]\\
&=\frac{1}{N}\left(\frac{1}{M_2}E\left[\frac{f_t^2(\bar{X}_{i_1})}{\hat{q}_1^2(\bar{X}_{i_1})}|\bar{\mathbf{X}}\right]+\left(1-\frac{1}{M_2}\right)E\left[\frac{f_t^2(\bar{X}_{i_1})}{\hat{q}_1(\bar{X}_{i_1})\hat{q}_2(\bar{X}_{i_1})}|\bar{\mathbf{X}}\right]E\left[\frac{\hat{q}_2(\bar{X}_{i_1})}{\hat{q}_1(\bar{X}_{i_1})}|\bar{\mathbf{X}}\right]\right)+\left(1-\frac{1}{N}\right)\left(\frac{1}{M_2}E\left[\frac{f_t^2(\bar{X}_{i_j})}{\hat{q}_1^2(\bar{X}_{i_j})}|\bar{\mathbf{X}}\right]+\left(1-\frac{1}{M_2}\right)E\left[\frac{f_t(\bar{X}_{i_j})}{\hat{q}_1(\bar{X}_{i_j})}|\bar{\mathbf{X}}\right]^2\right)\\
&=\frac{1}{M_2}E\left[\frac{f_t^2(\bar{X}_{i_1})}{\hat{q}_1^2(\bar{X}_{i_1})}|\bar{\mathbf{X}}\right]+\frac{1}{N}\left(1-\frac{1}{M_2}\right)E\left[\frac{f_t^2(\bar{X}_{i_1})}{\hat{q}_1(\bar{X}_{i_1})\hat{q}_2(\bar{X}_{i_1})}|\bar{\mathbf{X}}\right]E\left[\frac{\hat{q}_2(\bar{X}_{i_1})}{\hat{q}_1(\bar{X}_{i_1})}|\bar{\mathbf{X}}\right]+\left(1-\frac{1}{N}\right)\left(1-\frac{1}{M_2}\right)E\left[\frac{f_t(\bar{X}_{i_1})}{\hat{q}_1(\bar{X}_{i_1})}|\bar{\mathbf{X}}\right]^2,
\end{aligned}$$

where we use the following identity:

$$E\left[\left(\frac{1}{M}\sum_{s\geq 1}\sum_{i=1}^{M}f(X_{s,i})\right)\left(\frac{1}{M}\sum_{s\geq 1}\sum_{i=1}^{M}g(X_{s,i})\right)\right]=\frac{1}{M}\sum_{s\geq 1}E\left[f(X_{s,1})g(X_{s,1})\right]-\frac{1}{M}\sum_{s\geq 1}E\left[f(X_{s,1})\right]E\left[g(X_{s,1})\right]+\left(\sum_{s\geq 1}E\left[f(X_{s,1})\right]\right)\left(\sum_{s\geq 1}E\left[g(X_{s,1})\right]\right).$$

By expanding each expected value over all proposals in the set $\bar{\mathbf{X}}$, the above equation is expressed as:

$$\frac{1}{M_2}\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{f_t^2(\bar{X}_{s,i})}{\hat{q}_1^2(\bar{X}_{s,i})}\frac{\hat{q}_1(\bar{X}_{s,i})/p(\bar{X}_{s,i})}{\sum_{s\geq 1}\sum_{j=1}^{M_1}\hat{q}_1(\bar{X}_{s,j})/p(\bar{X}_{s,j})}+\frac{1}{N}\left(1-\frac{1}{M_2}\right)\left(\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{f_t^2(\bar{X}_{i_1})}{\hat{q}_1(\bar{X}_{i_1})\hat{q}_2(\bar{X}_{i_1})}\frac{\hat{q}_1(\bar{X}_{s,i})/p(\bar{X}_{s,i})}{\sum_{s\geq 1}\sum_{j=1}^{M_1}\hat{q}_1(\bar{X}_{s,i})/p(\bar{X}_{s,i})}\right)$$

$$\times\left(\sum_{s\geq 1}\sum_{j=1}^{M_1}\frac{\hat{q}_2(\bar{X}_{i_j})}{\hat{q}_1(\bar{X}_{i_j})}\frac{\hat{q}_1(\bar{X}_{s,i})/p(\bar{X}_{s,i})}{\sum_{s\geq 1}\sum_{j=1}^{M_1}\hat{q}_1(\bar{X}_{s,j})/p(\bar{X}_{s,j})}\right)+\left(1-\frac{1}{N}\right)\left(1-\frac{1}{M_2}\right)\left(\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{f_t(\bar{X}_{i_1})}{\hat{q}_1(\bar{X}_{i_1})}\frac{\hat{q}_1(\bar{X}_{s,i})/p(\bar{X}_{s,i})}{\sum_{s\geq 1}\sum_{j=1}^{M_1}\hat{q}_1(\bar{X}_{s,j})/p(\bar{X}_{s,j})}\right)^2.$$

$$=\frac{1}{M_2}\left(\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{f_t^2(\bar{X}_{s,i})}{\hat{q}_1(\bar{X}_{s,i})p(\bar{X}_{s,i})}\right)\left(\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{\hat{q}_1(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)^{-1}+\frac{1}{N}\left(1-\frac{1}{M_2}\right)\left(\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{f_t^2(\bar{X}_{s,i})}{p(\bar{X}_{s,i})\hat{q}_2(\bar{X}_{s,i})}\right)\left(\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{\hat{q}_2(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)\left(\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{\hat{q}_1(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)^{-2}$$

$$+\left(1-\frac{1}{N}\right)\left(1-\frac{1}{M_2}\right)\left(\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{f_t(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)^2\left(\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{\hat{q}_1(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)^{-2}.$$

Substituting the above equation, $E[\langle I_t^2\rangle_{\text{tsr}}^N|\bar{Z}_t]$ is expressed by:

$$\frac{1}{M_2}E\left[\left(\frac{1}{M_1}\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{f_t^2(\bar{X}_{s,i})}{\hat{q}_1(\bar{X}_{s,i})p(\bar{X}_{s,i})}\right)\left(\frac{1}{M_1}\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{\hat{q}_1(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)|\bar{Z}_t\right]+\frac{1}{N}\left(1-\frac{1}{M_2}\right)E\left[\left(\frac{1}{M_1}\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{f_t^2(\bar{X}_{s,i})}{p(\bar{X}_{s,i})\hat{q}_2(\bar{X}_{s,i})}\right)\left(\frac{1}{M_1}\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{\hat{q}_2(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)|\bar{Z}_t\right]$$

$$+\left(1-\frac{1}{N}\right)\left(1-\frac{1}{M_2}\right)E\left[\left(\frac{1}{M_1}\sum_{s\geq 1}\sum_{i=1}^{M_1}\frac{f_t(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)^2|\bar{Z}_t\right].$$

Then $E[\langle I_t^2\rangle_{\text{tsr}}^N|\bar{Z}_t]$ is expressed by (we omit the arguments for brevity):

$$\frac{1}{M_2}\left(\frac{1}{M_1}\sum_{s\geq 1}E\left[\frac{f_t^2}{p^2}|\bar{Z}_t\right]-\frac{1}{M_1}\sum_{s\geq 1}E\left[\frac{f_t^2}{p\hat{q}_1}|\bar{Z}_t\right]E\left[\frac{\hat{q}_1}{p}|\bar{Z}_t\right]+\left(\sum_{s\geq 1}E\left[\frac{f_t^2}{p\hat{q}_1}|\bar{Z}_t\right]\right)\left(\sum_{s\geq 1}E\left[\frac{\hat{q}_1}{p}|\bar{Z}_t\right]\right)\right)$$

$$+\frac{1}{N}\left(1-\frac{1}{M_2}\right)\left(\frac{1}{M_1}\sum_{s\geq 1}E\left[\frac{f_t^2}{p^2}|\bar{Z}_t\right]-\frac{1}{M_1}\sum_{s\geq 1}E\left[\frac{f_t^2}{p\hat{q}_2}|\bar{Z}_t\right]E\left[\frac{\hat{q}_2}{p}|\bar{Z}_t\right]+\left(\sum_{s\geq 1}E\left[\frac{f_t^2}{p\hat{q}_2}|\bar{Z}_t\right]\right)\left(\sum_{s\geq 1}E\left[\frac{\hat{q}_2}{p}|\bar{Z}_t\right]\right)\right)$$

$$+\left(1-\frac{1}{N}\right)\left(1-\frac{1}{M_2}\right)\left(\frac{1}{M_1}E\left[\frac{f_t^2}{p^2}|\bar{Z}_t\right]-\frac{1}{M_1}\sum_{s\geq 1}E\left[\frac{f_t}{p}|\bar{Z}_t\right]^2+\left(\sum_{s\geq 1}E\left[\frac{f_t^2}{p\hat{q}_2}|\bar{Z}_t\right]\right)^2\right).$$

By rearranging the above equation, the conditional expected value $E[\langle I_t^2\rangle_{\text{tsr}}^N|\bar{Z}_t]$ is expressed by:

$$E[\langle I_t^2\rangle_{\text{tsr}}^N|\bar{Z}_t]=\frac{1}{M_1}\sum_{s\geq 1}E\left[\frac{f_t^2}{p^2}|\bar{Z}_t\right]-\frac{1}{M_1M_2}\sum_{s\geq 1}E\left[\frac{f_t^2}{p\hat{q}_1}|\bar{Z}_t\right]E\left[\frac{\hat{q}_1}{p}|\bar{Z}_t\right]+\frac{1}{M_2}\left(\sum_{s\geq 1}E\left[\frac{f_t^2}{p\hat{q}_1}|\bar{Z}_t\right]\right)\left(\sum_{s\geq 1}E\left[\frac{\hat{q}_1}{p}|\bar{Z}_t\right]\right)$$

$$-\frac{1}{N}\frac{1}{M_1}\left(1-\frac{1}{M_2}\right)\sum_{s\geq 1}E\left[\frac{f_t^2}{p\hat{q}_2}|\bar{Z}_t\right]E\left[\frac{\hat{q}_2}{p}|\bar{Z}_t\right]+\frac{1}{N}\left(1-\frac{1}{M_2}\right)\left(\sum_{s\geq 1}E\left[\frac{f_t^2}{p\hat{q}_2}|\bar{Z}_t\right]\right)\left(\sum_{s\geq 1}E\left[\frac{\hat{q}_2}{p}|\bar{Z}_t\right]\right)$$

$$-\left(1-\frac{1}{N}\right)\frac{1}{M_1}\left(1-\frac{1}{M_2}\right)\sum_{s\geq 1}E\left[\frac{f_t}{p}|\bar{Z}_t\right]^2+\left(1-\frac{1}{N}\right)\left(1-\frac{1}{M_2}\right)\left(\sum_{s\geq 1}E\left[\frac{f_t}{p}|\bar{Z}_t\right]\right)^2.$$

To transform the above equation into the form of the variance, we expand the coefficients of the last line as follows:

$$-\left(1-\frac{1}{N}\right)\frac{1}{M_1}\left(1-\frac{1}{M_2}\right)\sum_{s\geq 1}E\left[\frac{f_t}{p}|\bar{Z}_t\right]^2+\left(1-\frac{1}{N}\right)\left(1-\frac{1}{M_2}\right)\left(\sum_{s\geq 1}E\left[\frac{f_t}{p}|\bar{Z}_t\right]\right)^2$$

$$=-\left(1-\frac{1}{N}-\frac{1}{M_2}+\frac{1}{N}\frac{1}{M_2}\right)\frac{1}{M_1}\sum_{s\geq 1}E\left[\frac{f_t}{p}|\bar{Z}_t\right]^2+\left(1-\frac{1}{N}-\frac{1}{M_2}+\frac{1}{N}\frac{1}{M_2}\right)\left(\sum_{s\geq 1}E\left[\frac{f_t}{p}|\bar{Z}_t\right]\right)^2$$

$$=-\frac{1}{M_1}\sum_{s\geq 1}E\left[\frac{f_t}{p}|\bar{Z}_t\right]^2+\frac{1}{M_1M_2}\sum_{s\geq 1}E\left[\frac{f_t}{p}|\bar{Z}_t\right]^2+\frac{1}{N}\frac{1}{M_1}\left(1-\frac{1}{M_2}\right)\sum_{s\geq 1}E\left[\frac{f_t}{p}|\bar{Z}_t\right]^2$$

$$+\left(\sum_{s\geq 1}E\left[\frac{f_t}{p}|\bar{Z}_t\right]\right)^2-\frac{1}{M_2}\left(\sum_{s\geq 1}E\left[\frac{f_t}{p}|\bar{Z}_t\right]\right)^2-\frac{1}{N}\left(1-\frac{1}{M_2}\right)\left(\sum_{s\geq 1}E\left[\frac{f_t}{p}|\bar{Z}_t\right]\right)^2.$$

2.3. Derivation of the conditional variance $V[\langle I_t\rangle_{\text{tsr}}^N|\bar{Z}_t]$

The conditional variance $V[\langle I_t\rangle_{\text{tsr}}^N|\bar{Z}_t]$ is calculated by subtracting $E[\langle I_t\rangle_{\text{tsr}}^N|\bar{Z}_t]^2 = (\sum_{s\geq 1} E[f_t/p|\bar{Z}_t])^2$ from the above equation. Rearranging each component leads to:

$$\frac{1}{M_1}\left(\sum_{s\geq 1} E\left[\frac{f_t^2}{p^2}|\bar{Z}_t\right] - \sum_{s\geq 1} E\left[\frac{f_t}{p}|\bar{Z}_t\right]^2\right) - \frac{1}{M_1 M_2}\left(\sum_{s\geq 1} E\left[\frac{f_t^2}{p\hat{q}_1}|\bar{Z}_t\right] E\left[\frac{\hat{q}_1}{p}|\bar{Z}_t\right] - \sum_{s\geq 1} E\left[\frac{f_t}{p}|\bar{Z}_t\right]^2\right)$$
$$+\frac{1}{M_2}\left(\left(\sum_{s\geq 1} E\left[\frac{f_t^2}{p\hat{q}_1}|\bar{Z}_t\right]\right)\left(\sum_{s\geq 1} E\left[\frac{\hat{q}_1}{p}|\bar{Z}_t\right]\right) - \left(\sum_{s\geq 1} E\left[\frac{f_t}{p}|\bar{Z}_t\right]^2\right)\right) - \frac{1}{N}\frac{1}{M_1}\left(1-\frac{1}{M_2}\right)\left(\sum_{s\geq 1} E\left[\frac{f_t^2}{p\hat{q}_2}|\bar{Z}_t\right] E\left[\frac{\hat{q}_2}{p}|\bar{Z}_t\right] - \sum_{s\geq 1} E\left[\frac{f_t}{p}|\bar{Z}_t\right]^2\right)$$
$$+\frac{1}{N}\left(1-\frac{1}{M_2}\right)\left(\left(\sum_{s\geq 1} E\left[\frac{f_t^2}{p\hat{q}_2}|\bar{Z}_t\right]\right)\left(\sum_{s\geq 1} E\left[\frac{\hat{q}_2}{p}|\bar{Z}_t\right]\right) - \left(\sum_{s\geq 1} E\left[\frac{f_t}{p}|\bar{Z}_t\right]^2\right)\right)$$

We now transform the product of two expected values shown in the above equation as:

$$E\left[\frac{f_t^2(\bar{X}_{s,1})}{p(\bar{X}_{s,1})\hat{q}_1(\bar{X}_{s,1})}|\bar{Z}_t\right] E\left[\frac{\hat{q}_1(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}|\bar{Z}_t\right] = \frac{1}{p^2(\bar{Z}_t)}\left(\int_{A^s}\frac{f_t^2(\bar{y}_s\bar{Z}_t)}{\hat{q}_1(\bar{y}_s\bar{Z}_t)}d\mu(\bar{y}_s)\right)\underbrace{\left(\int_{A^s}\hat{q}_1(\bar{y}_s\bar{Z}_t)d\mu(\bar{y}_s)\right)}_{Q_{1,s}} = \frac{1}{p^2(\bar{Z}_t)}\left(\int_{A^s}\frac{f_t^2(\bar{y}_s\bar{Z}_t)}{\hat{q}_1(\bar{y}_s\bar{Z}_t)/Q_{1,s}}d\mu(\bar{y}_s)\right)$$
$$= \frac{1}{p^2(\bar{Z}_t)}\left(\int_{A^s}\frac{f_t^2(\bar{y}_s\bar{Z}_t)}{q_{1,s}(\bar{y}_s|\bar{Z}_t)}d\mu(\bar{y}_s)\right) = \frac{1}{p^2(\bar{Z}_t)}E\left[\frac{f_t^2(\bar{X}_{s,1})}{q_{1,s}^2(\bar{Y}_{s,1}|\bar{Z}_t)}|\bar{Z}_t\right] = E\left[\frac{f_t^2(\bar{X}_{s,1})}{q_{1,s}^2(\bar{X}_{s,1})}|\bar{Z}_t\right],$$

where $Q_{1,s}$ is the normalization factor that is the integral of the target distribution $\hat{q}_1$ over all the light sub-paths $\bar{y}_s$ with s vertices, and $q_1(\bar{y}_s|\bar{Z}_t)$ is the conditional pdf, and $q_1(\bar{x}_s) = q_1(\bar{y}_s|\bar{Z}_t)p(\bar{Z}_t)$. Similary, the following product of two expected values can be calculated as:

$$\left(\sum_{s\geq 1} E\left[\frac{f_t^2}{p\hat{q}_1}|\bar{Z}_t\right]\right)\left(\sum_{s\geq 1} E\left[\frac{\hat{q}_1}{p}|\bar{Z}_t\right]\right) = \frac{1}{p^2(\bar{Z}_t)}\left(\int_{\mathcal{A}}\frac{f_t^2(\bar{y}\bar{Z}_t)}{\hat{q}_1(\bar{y}\bar{Z}_t)}d\mu(\bar{y})\right)\underbrace{\left(\int_{\mathcal{A}}\hat{q}_1(\bar{y}\bar{Z}_t)d\mu(\bar{y})\right)}_{Q_1} = \frac{1}{p^2(\bar{Z}_t)}\left(\int_{\mathcal{A}}\frac{f_t^2(\bar{y}\bar{Z}_t)}{\hat{q}_1(\bar{y}\bar{Z}_t)/Q_1}d\mu(\bar{y})\right)$$
$$= \frac{1}{p^2(\bar{Z}_t)}\left(\int_{\mathcal{A}}\frac{f_t^2(\bar{y}\bar{Z}_t)}{q_1(\bar{y}_s|\bar{Z}_t)}d\mu(\bar{y})\right) = E\left[\frac{f_t^2}{q_1^2}|\bar{Z}_t\right],$$

where Q_1 is the normalization factor that is the integral of the target distribution $\hat{q}_1$ over all the light sub-paths $\bar{y}_s$ with arbitrary length ($s \geq 1$), and $q_1(\bar{y}|\bar{Z}_t)$ is the conditional pdf, and $q_1(\bar{x}) = q_1(\bar{y}|\bar{Z}_t)p(\bar{Z}_t)$. The integral domain $\mathcal{A}$ is a union of A^s as $\mathcal{A} = \cup_{s\geq 1}A^s$. In the similar way, the following relations hold for pdfs $q_{2,s}$ and q_2, which are defined similarly as $q_{2,s}(\bar{x}) = q_{2,s}(\bar{y}_s|\bar{Z}_t)p(\bar{Z}_t)$ and $q_2(\bar{x}) = q_2(\bar{y}|\bar{Z}_t)p(\bar{Z}_t)$

$$E\left[\frac{f_t^2}{p\hat{q}_2}|\bar{Z}_t\right] E\left[\frac{\hat{q}_2}{p}|\bar{Z}_t\right] = E\left[\frac{f_t^2}{q_{2,s}^2}|\bar{Z}_t\right], \quad \left(\sum_{s\geq 1} E\left[\frac{f_t^2}{p\hat{q}_2}|\bar{Z}_t\right]\right)\left(\sum_{s\geq 1} E\left[\frac{\hat{q}_2}{p}|\bar{Z}_t\right]\right) = E\left[\frac{f_t^2}{q_2^2}|\bar{Z}_t\right],$$

Then the conditional variance $V[\langle I_t\rangle_{\text{tsr}}^N|\bar{Z}_t]$is expressed by:

$$\frac{1}{M_1}\sum_{s\geq 1} V\left[\frac{f_t}{p}|\bar{Z}_t\right] - \frac{1}{M_1 M_2}\sum_{s\geq 1} V\left[\frac{f_t}{q_{1,s}}|\bar{Z}_t\right] + \frac{1}{M_2}V\left[\frac{f_t}{q_1}|\bar{Z}_t\right] - \frac{1}{N}\frac{1}{M_1}\left(1-\frac{1}{M_2}\right)\sum_{s\geq 1} V\left[\frac{f_t}{q_{s,2}}|\bar{Z}_t\right] + \frac{1}{N}\left(1-\frac{1}{M_2}\right)V\left[\frac{f_t}{q_2}|\bar{Z}_t\right]. \tag{11}$$

2.4. Derivation of the variance $V[\langle I_t\rangle_{\text{tsr}}^N]$ for two-stage resampling

We now derive the variance $V[\langle I_t\rangle_{\text{tsr}}^N]$ by taking into accout the randomness of the eye sub-path $\bar{Z}_t$. The variance $V[\langle I_t\rangle_{\text{tsr}}^N]$ is calculated by the law of total variance as:

$$V[\langle I_t\rangle_{\text{tsr}}^N] = E[V[\langle I_t\rangle_{\text{tsr}}^N|\bar{Z}_t]] + V[E[\langle I_t\rangle_{\text{tsr}}^N|\bar{Z}_t]]. \tag{12}$$

So far, we have included the pdf $p(\bar{Z}_t)$ in the pdfs p, $q_{1,s}$, $q_{2,s}$, q_1, and q_2. To take into account the randomness of the eye sub-path $\bar{Z}_t$, we decompose the pdfs into the pdf $p(\bar{Z}_t)$ and the (conditional) pdfs as:

$$p(\bar{x}_s) = p(\bar{y}_s)p(\bar{Z}_t), q_{1,s}(\bar{x}_s) = q_{1,s}(\bar{y}_s|\bar{Z}_t)p(\bar{Z}_t), q_1(\bar{x}) = q_1(\bar{y}|\bar{Z}_t)p(\bar{Z}_t).$$

By using $E[V[a|b]b^2] = V[ab] - V[E[a|b]b]$ and considering $1/p(\bar{Z}_t)$ as random variables, the following expected value of variance is expressed by:

$$E\left[V\left[\frac{f_t(\bar{X}_s)}{p(\bar{X}_s)}|\bar{Z}_t\right]\right] = E\left[V\left[\frac{f_t(\bar{X}_s)}{p(\bar{Y}_s)}|\bar{Z}_t\right]\frac{1}{p^2(\bar{Z}_t)}\right] = V\left[\frac{f_t(\bar{X}_s)}{p(\bar{Z}_t)p(\bar{Y}_s)}\right] - V\left[E\left[\frac{f_t(\bar{X}_s)}{p(\bar{Y}_s)}|\bar{Z}_t\right]\frac{1}{p(\bar{Z}_t)}\right] = V\left[\frac{f_t(\bar{X}_s)}{p(\bar{X}_s)}\right] - V\left[\frac{1}{p(\bar{Z}_t)}\int_{A^s} f_t(\bar{y}_s\bar{Z}_t)d\mu(\bar{y}_s)\right].$$

The expected value of each variance term in Eq. (10) can be rewritten in the similar way. Then $E[V[\langle I_t\rangle_{\text{tsr}}^N|\bar{Z}_t]]$ is expressed by:

$$\frac{1}{M_1}\sum_{s\geq 1}\left(V\left[\frac{f_t}{p}\right]-V\left[\frac{1}{p(\bar{Z}_t)}\int_{A^s} f_t(\bar{x}_s)d\mu(\bar{y}_s)\right]\right)-\frac{1}{M_1M_2}\sum_{s\geq 1}\left(V\left[\frac{f_t}{q_{1,s}}\right]-V\left[\frac{1}{p(\bar{Z}_t)}\int_{A^s} f_t(\bar{x}_s)d\mu(\bar{y}_s)\right]\right)$$
$$+\frac{1}{M_2}\left(V\left[\frac{f_t}{q_1}\right]-V\left[\frac{1}{p(\bar{Z}_t)}\int_{\mathcal{A}} f_t(\bar{x})d\mu(\bar{y})\right]\right)-\frac{1}{N}\frac{1}{M_1}\left(1-\frac{1}{M_2}\right)\sum_{s\geq 1}\left(V\left[\frac{f_t}{q_{s,2}}\right]-V\left[\frac{1}{p(\bar{Z}_t)}\int_{A^s} f_t(\bar{x}_s)d\mu(\bar{y}_s)\right]\right)$$
$$+\frac{1}{N}\left(1-\frac{1}{M_2}\right)\left(V\left[\frac{f_t}{q_2}\right]-V\left[\frac{1}{p(\bar{Z}_t)}\int_{\mathcal{A}} f_t(\bar{x})d\mu(\bar{y})\right]\right).$$

By rearranging the above equation and adding $V[E[\langle I_t\rangle_{\text{tsr}}^N|\bar{Z}_t]]$, the variance $V[\langle I_t\rangle_{\text{tsr}}^N]$ is represented by:

$$\frac{1}{M_1}\sum_{s\geq 1}\left(V\left[\frac{f_t}{p}\right]-V\left[\frac{1}{p(\bar{Z}_t)}\int_{A^s} f_t(\bar{x}_s)d\mu(\bar{y}_s)\right]\right)-\frac{1}{M_1M_2}\sum_{s\geq 1}\left(V\left[\frac{f_t}{q_{1,s}}\right]-V\left[\frac{1}{p(\bar{Z}_t)}\int_{A^s} f_t(\bar{x}_s)d\mu(\bar{y}_s)\right]\right)$$
$$+\frac{1}{M_2}\left(V\left[\frac{f_t}{q_1}\right]-V\left[\frac{1}{p(\bar{Z}_t)}\int_{\mathcal{A}} f_t(\bar{x})d\mu(\bar{y})\right]\right)-\frac{1}{N}\frac{1}{M_1}\left(1-\frac{1}{M_2}\right)\sum_{s\geq 1}\left(V\left[\frac{f_t}{q_{s,2}}\right]-V\left[\frac{1}{p(\bar{Z}_t)}\int_{A^s} f_t(\bar{x}_s)d\mu(\bar{y}_s)\right]\right)$$
$$+\frac{1}{N}\left(1-\frac{1}{M_2}\right)\left(V\left[\frac{f_t}{q_2}\right]-V\left[\frac{1}{p(\bar{Z}_t)}\int_{\mathcal{A}} f_t(\bar{x})d\mu(\bar{y})\right]\right)+\underbrace{V\left[\frac{1}{p(\bar{Z}_t)}\int_{\mathcal{A}} f_t(\bar{x})d\mu(\bar{y})\right]}_{V[E[\langle I_t\rangle_{\text{tsr}}^N|\bar{Z}_t]]}$$
$$=\frac{1}{M_1}\sum_{s\geq 1}V\left[\frac{f_t}{p}\right]-\frac{1}{M_1M_2}\sum_{s\geq 1}V\left[\frac{f_t}{q_{1,s}}\right]+\frac{1}{M_2}V\left[\frac{f_t}{q_1}\right]-\frac{1}{N}\frac{1}{M_1}\left(1-\frac{1}{M_2}\right)\sum_{s\geq 1}V\left[\frac{f_t}{q_{s,2}}\right]+\frac{1}{N}\left(1-\frac{1}{M_2}\right)V\left[\frac{f_t}{q_2}\right]$$
$$-\frac{1}{M_1}\left(1-\frac{1}{M_2}\right)\left(1-\frac{1}{N}\right)\sum_{s\geq 1}V\left[\frac{1}{p(\bar{Z}_t)}\int_{A^s} f_t(\bar{x}_s)d\mu(\bar{y}_s)\right]+\left(1-\frac{1}{N}\right)\left(1-\frac{1}{M_2}\right)V\left[\frac{1}{p(\bar{Z}_t)}\int_{\mathcal{A}} f_t(\bar{x})d\mu(\bar{y})\right].$$

In summary, the variance of the (N-sample) two-stage resampling estimator $V[\langle I_t\rangle_{\text{tsr}}^N]$ for the resampling strategy t is represented by:

$$V[\langle I_t\rangle_{\text{tsr}}^N]=\frac{1}{M_1}\sum_{s\geq 1}V\left[\frac{f_t}{p}\right]-\frac{1}{M_1M_2}\sum_{s\geq 1}V\left[\frac{f_t}{q_{1,s}}\right]+\frac{1}{M_2}V\left[\frac{f_t}{q_1}\right]-\frac{1}{N}\frac{1}{M_1}\left(1-\frac{1}{M_2}\right)\sum_{s\geq 1}V\left[\frac{f_t}{q_{s,2}}\right]+\frac{1}{N}\left(1-\frac{1}{M_2}\right)V\left[\frac{f_t}{q_2}\right]$$
$$-\frac{1}{M_1}\left(1-\frac{1}{M_2}\right)\left(1-\frac{1}{N}\right)\sum_{s\geq 1}V\left[\frac{1}{p(\bar{Z}_t)}\int_{A^s} f_t(\bar{x}_s)d\mu(\bar{y}_s)\right]+\left(1-\frac{1}{N}\right)\left(1-\frac{1}{M_2}\right)V\left[\frac{1}{p(\bar{Z}_t)}\int_{\mathcal{A}} f_t(\bar{x})d\mu(\bar{y})\right]. \quad (13)$$

The second line in Eq. (13) includes the variance terms of integrals, which makes the minimization of the upper bound of the variance $V[\langle I_t\rangle_{\text{tsr}}^N]$ infeasible. To address this problem, we set the number of samples to one ($N=1$) to vanish the coefficients for the variance terms including the integrals as:

$$V[\langle I_t\rangle_{\text{tsr}}^1]=\frac{1}{M_1}\sum_{s\geq 1}V\left[\frac{f_t}{p}\right]-\frac{1}{M_1M_2}\sum_{s\geq 1}V\left[\frac{f_t}{q_{1,s}}\right]+\frac{1}{M_2}V\left[\frac{f_t}{q_1}\right]-\frac{1}{M_1}\left(1-\frac{1}{M_2}\right)\sum_{s\geq 1}V\left[\frac{f_t}{q_{s,2}}\right]+\left(1-\frac{1}{M_2}\right)V\left[\frac{f_t}{q_2}\right]. \quad (14)$$

3. Derivation of weighting functions w_t for two-stage resampling

3.1. The variance of the pixel measurement $V[\langle I\rangle]$

We now derive the weighting functions by minimizing the upper bound of the variance $V[\langle I\rangle]$ of the pixel measurement I. For a light path $\bar{x}$ with length k, the variance $V[\langle I\rangle]$ is calculated by the sum of variances for $k+2$ strategies as:

$$V[\langle I\rangle]=\sum_{t\in\Lambda_{\text{tsr}}}V[\langle I_t\rangle_{\text{tsr}}^1]+\frac{1}{N_t}\sum_{t\in\Lambda_{\text{is}}}V\left[\frac{f_t}{p}\right],$$

where $\Lambda_{tsr}\in\{2,\ldots,k\}$ represents the resampling strategies and $\Lambda_{is}\in\{0,1,k+1\}$ represents the strategies handled by BPT, and N_t is the number of samples for BPT. By substituting $V[\langle I_t\rangle_{\text{tsr}}^1]$ in Eq. (14), $V[\langle I\rangle]$ is expressed by:

$$V[\langle I\rangle]=\sum_{t\in\Lambda_{\text{tsr}}}\left(\frac{1}{M_1}V\left[\frac{f_t}{p}\right]-\frac{1}{M_1M_2}V\left[\frac{f_t}{q_{1,s}}\right]+\frac{1}{M_2}V\left[\frac{f_t}{q_1}\right]-\frac{1}{M_1}\left(1-\frac{1}{M_2}\right)V\left[\frac{f_t}{q_{s,2}}\right]+\left(1-\frac{1}{M_2}\right)V\left[\frac{f_t}{q_2}\right]\right)+\frac{1}{N_t}\sum_{t\in\Lambda_{\text{is}}}V\left[\frac{f_t}{p}\right], \quad (15)$$

where the summation over s is eliminated since the number of the light sub-path vertices is uniquely determined as $s=k+1-t$.

3.2. Derivation of the weighting functions

We derive the weighting functions that minimize the upper bound of the variance $V[\langle I\rangle] = E[\langle I^2\rangle] - E[\langle I\rangle]^2$. Specifically, we aim to minimize the second moment $E[\langle I^2\rangle]$ as:

$$E[\langle I^2\rangle] = \sum_{t\in\Lambda_{\mathrm{tsr}}}\left(\frac{1}{M_1}E\left[\frac{f_t^2}{p^2}\right] - \frac{1}{M_1M_2}E\left[\frac{f_t^2}{q_{1,s}^2}\right] + \frac{1}{M_2}E\left[\frac{f_t^2}{q_1^2}\right] - \frac{1}{M_1}\left(1-\frac{1}{M_2}\right)E\left[\frac{f_t^2}{q_{s,2}^2}\right] + \left(1-\frac{1}{M_2}\right)E\left[\frac{f_t^2}{q_2^2}\right]\right) + \frac{1}{N_t}\sum_{t\in\Lambda_{\mathrm{is}}}E\left[\frac{f_t^2}{p^2}\right].$$

We represent each expected value term with the integral form as $E[f^2/p^2] = \int f^2(\bar{x})/p(\bar{x})d\mu(\bar{x})$. Since it is sufficient to minimize the integrand at each path $\bar{x}$ separately and $f(\bar{x})$ is constant for all strategies, we minimize the following objective function subject to the condition $\sum_{t=0}^{k+1} w_t = 1$:

$$\sum_{t\in\Lambda_{\mathrm{tsr}}}\left(\frac{1}{M_1}\frac{w_t^2}{p} - \frac{1}{M_1M_2}\frac{w_t^2}{q_{1,s}} + \frac{1}{M_2}\frac{w_t^2}{q_1} - \frac{1}{M_1}\left(1-\frac{1}{M_2}\right)\frac{w_t^2}{q_{s,2}} + \left(1-\frac{1}{M_2}\right)\frac{w_t^2}{q_2}\right) + \frac{1}{N_t}\sum_{t\in\Lambda_{\mathrm{is}}}\frac{w_t^2}{p}.$$

In the above equation, we represent the pdfs $q_{1,s}$ and $q_{2,s}$ with q_1 and q_2 as:

$$q_{1,s}(\bar{x}) = \frac{\hat{q}_1(\bar{x})}{\int_{A^s}\hat{q}_1(\bar{x})d\mu(\bar{y}_s)} = \frac{\hat{q}_1(\bar{x})}{\int_{\mathcal{A}}\hat{q}_1(\bar{x})d\mu(\bar{y})}\cdot\frac{\int_{\mathcal{A}}\hat{q}_1(\bar{x})d\mu(\bar{y})}{\int_{A^s}\hat{q}_1(\bar{x})d\mu(\bar{y}_s)} = q_1(\bar{x})\frac{Q_1}{Q_{1,s}}, \tag{16}$$

where $Q_{1,s}$ is the normalization factor of $\hat{q}_{1,s}$ over A^s, and Q_1 is that of $\hat{q}_1$ over $\mathcal{A}$. Similarly, $q_{2,s}(\bar{x}) = q_2(\bar{x})Q_2/Q_{2,s}$. By using these, the objective function is represented as:

$$\sum_{t\in\Lambda_{tsr}}\left(\frac{1}{M_1}\frac{1}{p(\bar{x})} + \left(1-\frac{1}{M_1}\frac{Q_{1,s}}{Q_1}\right)\frac{1}{M_2}\frac{1}{q_1(\bar{x})} + \left(1-\frac{1}{M_1}\frac{Q_{2,s}}{Q_2}\right)\left(1-\frac{1}{M_2}\right)\frac{1}{q_2(\bar{x})}\right)w_t^2(\bar{x}) + \sum_{t\in\Lambda_{is}}\frac{1}{N_t}\frac{w_t(\bar{x})^2}{p(\bar{x})}.$$

Although it is possible to estimate $Q_{1,s}$ and $Q_{2,s}$ by using Monte Carlo integration, it is difficult to keep the normalization factors $Q_{1,s}$ and $Q_{2,s}$ for all the non-negative integers s. Fortunately, we can bound the ratios $Q_{1,s}/Q$ and $Q_{2,s}/Q$ between zero and one by definition, and we approximate the ratios $Q_{1,s}/Q_1$ and $Q_{2,s}/Q_2$ with the upper bound one. Then we define the following density p_{tsr} as:

$$p_{\mathrm{tsr}}(\bar{x}) = \left(\frac{1}{M_1}\frac{1}{p(\bar{x})} + \left(1-\frac{1}{M_1}\right)\left(\frac{1}{M_2}\frac{1}{q_1(\bar{x})} + \left(1-\frac{1}{M_2}\right)\frac{1}{q_2(\bar{x})}\right)\right)^{-1}.$$

By using the density p_{tsr}, the objective function is simplified to $\sum_{t=0}^{k+1} n_t p_t(\bar{x})$ where $n_t = 1$ and $p_t(\bar{x}) = p_{\mathrm{tsr}}(\bar{x})$ for $t \in \Lambda_{tsr}$ and $n_t = N_t$ for $t \in \Lambda_{is}$. This makes it possible to derive the weighting function using the balance heuristic as:

$$w_t(\bar{x}) = \frac{n_t p_t(\bar{x})}{\sum_{i=0}^{k+1} n_i p_i(\bar{x})}, \tag{17}$$

$$n_t = \begin{cases} 1 & (t\in\Lambda_{tsr}) \\ N_t & (t\in\Lambda_{is}) \end{cases}, \tag{18}$$

$$p_t(\bar{x}) = \begin{cases} \left(\frac{1}{M_1}\frac{1}{p(\bar{x})} + \left(1-\frac{1}{M_1}\right)\left(\frac{1}{M_2}\frac{1}{q_1(\bar{x})} + \left(1-\frac{1}{M_2}\right)\frac{1}{q_2(\bar{x})}\right)\right)^{-1} & (t\in\Lambda_{tsr}) \\ p(\bar{x}) & (t\in\Lambda_{is}). \end{cases} \tag{19}$$